

SPECIAL VALUES OF PARTIAL ZETA FUNCTIONS OF REAL QUADRATIC FIELDS AT NONPOSITIVE INTEGERS AND EULER-MACLAURIN FORMULA

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ABSTRACT. We compute the special values at nonpositive integers of the partial zeta function of an ideal of a real quadratic field applying an asymptotic version of Euler-Maclaurin formula to the lattice cone associated to the ideal considered. The Euler-Maclaurin formula involved is obtained by applying the Todd series of differential operators to an integral of a small perturbation of the cone. The additive property of Todd series w.r.t. the cone decomposition enables us to express the partial zeta values in terms of the continued fraction of the reduced element of the ideal. The expression obtained uses the positive continued fraction which yields a virtual decomposition of the cone.

We apply the expression to some indexed families of real quadratic fields satisfying certain condition on the shape of the continued fractions. The families considered include those appeared in [20] and [21] as well as the Richaud-Degert types. We show that the partial zeta values at a given nonpositive integer $-k$ in the family indexed by n is a polynomial of n .

Finally, we compute explicitly the polynomials producing the partial zeta values at $s = -k$ for small k of some chosen families and compare these with some previously known results.

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Date: 2012.9.18.

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1. INTRODUCTION

Let K be a number field of the extension degree $[K : \mathbb{Q}] = r_1 + 2r_2$, where r_1 and r_2 denote respectively the number of real and complex embeddings of K . The Dedekind zeta function

$$\zeta_K(s) = \prod_{\mathfrak{p}:\text{prime ideal in } K} \frac{1}{1 - N\mathfrak{p}^{-s}}$$

is encoded with many interesting arithmetic properties of K . In particular, the residue at $s = 1$ is associated to the class number h_K of K by the class number formula:

$$\text{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} R_K h_K}{\omega_K \sqrt{|D_K|}},$$

where R_K is the regulator, ω_K is the number of roots of 1 in K and D_K is the discriminant. This has been the starting point of most studies of class numbers.

The simplest is the case of imaginary quadratic fields where the regulator appears to be trivial. In [13], Gauss listed 9 imaginary quadratic fields of class number 1 and conjectured that the list is complete. Later on this had been studied through 20th century and is now quite well understood and solved by works of Heegner, Stark, Goldfeld and several others(eg. [1], [17], [31], [15], [16], [18], [31] and [32]).

The case of real quadratic fields is more complicated due to the presence of nontrivial regulator. It is also conjectured by Gauss that there are infinitely many real quadratic fields of class number one. But since

the regulator is far from being controlled in relation to the discriminant, and there has been no essential progress to the proof of the conjecture.

Instead of treating the whole real quadratic fields, people considered some families of real quadratic fields where the regulators are controlled in relation to the discriminant. The most well-known family of this kind is the Richaud-Degert type: A Richaud-Degert type is defined by

$$d(n) = n^2 \pm r$$

for $r|4n$ and $-n < r \leq n$. For r fixed as above, the family $\{K_n\}$ of real quadratic fields is called R-D type. In this case, we have a bound of the regulator R_{K_n} :

$$R_{K_n} < 3 \log \sqrt{D_{K_n}}$$

As in imaginary quadratic case, a well-known estimation of Siegel $L(1, \chi_D) \sim |D|^{-\epsilon}$ together with the class number formula implies that there are only finitely many R-D type fields of class number one. Assuming the generalized Riemann hypothesis, the class number one problems have been solved for many subfamilies in R-D type.

It is quite recent that Biró first obtained an Riemann hypothesis free answer to the class number one problem for the families $K_n = \mathbb{Q}(\sqrt{n^2 + 4})$ and $K_n = \mathbb{Q}(\sqrt{4n^2 + 1})$ in a series of papers([2], [3]). He investigated the behavior of the special values of the partial Hecke L-functions at $s = 0$ in the family. The partial Hecke L-function of an ideal \mathfrak{a} is defined for a ray class character χ as

$$L(s, \mathfrak{a}, \chi) := \sum_{\mathfrak{b} \sim \mathfrak{a}} \frac{\chi(\mathfrak{b})}{N\mathfrak{b}^s}.$$

He discovered that the special values behave in a packet of linear forms whose coefficients are easily computed for the family $(K_n, O_{K_n}, \chi_n := \chi \circ N_{K_n/\mathbb{Q}})$ for a Dirichlet character χ . This property is named the *linearity*.

Inspired by Biró's pioneering work, in [5], [6], [25] and [26] the linearity is observed for more general families of Richaud-Degert types and the class number one and two problems have been answered for these.

In [20], we found a sufficient condition to yield the linearity of the Hecke L-values at $s = 0$. Namely, for families of integral ideals $\{\mathfrak{b}_n$ in $K_n\}$ such that $\mathfrak{b}_n^{-1} = [1, \omega(n)] := \mathbb{Z}1 + \mathbb{Z}\omega(n)$, where $\omega(n)$ has purely periodic positive continued fraction expansion of a fixed period r

$$\omega(n) = [[a_0(n), a_1(n), \dots, a_{r-1}(n)]]$$

and $N(\mathfrak{b}_n)N(x\omega(n) + y) = b_0(n)x^2 + b_1(n)xy + b_2(n)y^2$ for $a_i(n)$ and $b_i(n)$ being integer coefficient linear forms in n . In this setting we have,

for $n = qk + r$ with $0 \leq r < q$ and for a Dirichlet character χ of conductor q , the L -value at $s = 0$

$$L_{K_n}(0, \chi \circ N_{K_n}, \mathfrak{b}_n) = \frac{1}{12q^2} (A_\chi(r)k + B_\chi(r))$$

with $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi]$ where $\mathbb{Z}[\chi]$ denotes the extension of \mathbb{Z} by the values of χ .

In [21] we obtained a higher degree generalization of the linearity for ray class partial zeta values. Let $\omega(n)$ be the Gauss' reduced element of \mathfrak{b}_n . If we allow the coefficient $a_i(n)$ of the continued fraction of $\omega(n)$ to be polynomial of degree d , then the partial zeta value at $s = 0$ of a mod- q ray class ideal $(C + D\omega(n))\mathfrak{b}_n$ in the class of \mathfrak{b}_n is a quasi-polynomial in n :

$$\zeta_q(0, (C + D\omega(n))\mathfrak{b}_n) = \frac{1}{12q^2} (A_0(r) + A_1(r)k + \cdots + A_d(r)k^d)$$

with $A_i(r) \in \mathbb{Z}$ (for precise definition, we refer the reader to *loc.cit.*). In particular, if we take $d = 1$ and sum the ray class zeta values twisted by χ_n , one can recover the linearity of the partial Hecke L -values. For $d > 1$, the same process concludes the polynomial behavior of the partial Hecke values at $s = 0$.

The purpose of this article is to generalize our earlier work to special values at every nonpositive integer of the ideal class partial zeta functions under the same assumption for the family (K_n, \mathfrak{b}_n) . We assume again $\mathfrak{b}_n^{-1} = [1, \omega(n)]$ where $\omega(n)$ has purely periodic continued fraction expansion of fixed period r :

$$\omega(n) = [[a_0(n), a_1(n), \dots, a_{r-1}(n)]]$$

such that and $N(\mathfrak{b}_n)N(x\omega(n) + y) = b_0(n)x^2 + b_1(n)xy + b_2(n)y^2$ for $a_i(n)$ and $b_i(n)$ being integer coefficient polynomials. For the next two theorems, let ℓ be the even period of $\omega(n)$ (hence independent of n and $\ell = 2r$ (reps. r) if r is odd (reps. even)).

Our main result in this paper is as follows:

Theorem 1.1. *Let N be a fixed subset of \mathbb{N} . Suppose (K_n, \mathfrak{b}_n) satisfies the above condition for every $n \in N$. Then the special value of the partial zeta function of \mathfrak{b}_n at $s = -k$ for $k = 0, 1, 2, \dots$, is given by a polynomial in n :*

$$\zeta_{K_n}(-k, \mathfrak{b}_n) = A_0 + A_1n + A_2n^2 + \dots + A_mn^m$$

of degree bounded by $m = kC + D$ with the coefficients $A_i \in \frac{1}{C_k}\mathbb{Z}$, where C , D and C_k are given as follows:

$$C = 2 \deg \alpha_{\ell-1} + \deg b_1$$

$$D = \max_{0 \leq i \leq r-1} \{\deg a_i\}$$

and

$C_k = \text{LCM of}$

$$\left\{ \text{the denominators of } \frac{B_{i+1}B_{2k+1-i}}{(i+1)(2k+1-i)} \text{ and } \frac{B_{2k+2}}{(2k+2)(2k+1)} \binom{2k}{i}^{-1} \right\}_{0 \leq i \leq k}.$$

Our main theorem is a direct consequence of the following estimation of the partial zeta values of an ideal \mathfrak{b} in a real quadratic field K .

Let \mathfrak{b} be an integral ideal such that $\mathfrak{b}^{-1} = [1, \omega]$ for $\omega > 1$ and $0 < \omega' < 1$. Let α_i, β_i are coordinates of some lattice vectors determined by (the continued fraction of) ω . In particular, $\alpha_{\ell-1}\omega + \beta_{\ell-1}1$ is the totally positive fundamental unit of $K = K_n$. See Sec.6 for details. As usual, B_i denotes the i -th Bernoulli number.

Theorem 1.2. *Let \mathfrak{b} be an ideal of a real quadratic field K such that $\mathfrak{b}^{-1} = [1, \omega]$ where $\omega = [[a_0, a_1, \dots, a_{r-1}]]$. Then we have*

$$\begin{aligned} \zeta(-k, \mathfrak{b}) = & \sum_{i=0}^{l-1} (-1)^{i-1} L_k(\partial_{h_1}, \partial_{h_2}) Q(\alpha_i h_1 - \alpha_{i-1} h_2, \beta_i h_1 - \beta_{i-1} h_2)^k \\ & + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{l-1} (-1)^i a_{\ell-i} R_k(\partial_{h_1}, \partial_{h_2}) Q(\alpha_{i-2} h_1 + \alpha_i h_2, \beta_{i-2} h_1 + \beta_i h_2)^k \end{aligned}$$

In the above, L_k and R_k are the homogeneous polynomials of degree $2k$:

$$(1.1) \quad L_k(X, Y) = \sum_{i=1}^{2k+1} \frac{B_i}{i!} \frac{B_{2k+2-i}}{(2k+2-i)!} X^{i-1} Y^{2k-i+1},$$

$$(1.2) \quad R_k(X, Y) = X^{2k} + X^{2k-1}Y + \dots + Y^{2k}.$$

It is not surprising that this behavior of the partial zeta or L-values is related to the pattern of the continued fractions in the family if we note that the Shintani cone decomposition arises in relation to the continued fraction. The significance of this fact lies on that for real quadratic fields, the regulator is controlled not only by the discriminant but also by the period r of the positive continued fraction of the reduced element ω in K . This is due to the following well-known upper bound:

$$R_K \leq r \log \sqrt{D_K}.$$

Study of special values of zeta or L-functions goes back to Euler. Euler evaluated $\zeta(-k)$ for $k = 0, 1, 2, \dots$, by using Euler-MacLaurin summation formula(cf. [8]). Later on, Siegel computed the values at non-positive integers of $\zeta_K(\mathfrak{b}, \mathfrak{f}, s)$ the ray class partial zeta function for an ideal \mathfrak{b} in a totally real number field K w.r.t. a conductor \mathfrak{f} based on the theory of modular forms([30]). Shintani established a combinatorial description of the zeta values at nonpositive integers([29]). Beside the complicated contour integral, Shintani's method is a reminiscence of Euler's. Similar approach was taken independently by Zagier in his evaluation of the partial zeta functions of real quadratic fields at non-positive integers([34]). Actually, the Shintani's method has a strength over Siegel's that it is ready to use in p-adic interpolation in case of totally real fields via Cartier duality. This view was clarified by Katz in [24].

Our evaluation is along with the line of Shintani and Zagier. We will apply a version of Euler-Maclaurin summation formula due to Karshon-Sternberg-Weitsman([22]). In *loc.cit.*, they made a version of Euler-Maclaurin formula taking care of the remainder term, so that one can apply this to expand asymptotically a function given as summation of exponentials. Since one side of the Euler-Maclaurin formula is application of appropriate version of Todd differential operator, the decomposition of Shintani cone is reflected additively due to the additivity of the Todd series under cone decomposition(See SS.4.4.).

Similar computation was done by Garoufalidis-Pommersheim([12]). They applied the Euler-Maclaurin summation formula of Brion-Vergne([4]) to obtain the asymptotic expansion. Brion-Vergne's formula is exact summation on the lattice points inside a simple polytope valid for polynomials or polynomials in exponentials of linear forms. They took the Shintani cone as the cone over a lattice polytope and varied the size inside the cone. In their treatment, the exact and the error terms are considered separately. The formula of Brion-Vergne is applied to the exact term and the error term is shown to be appropriately bounded.

Our method differs from that of Garoufalidis and Pommersheim in two directions: First, we used positive continued fraction while they took negative continued fractions. Basically, via the transition formula between positive and negative continued fractions, they contain more or less the same information of the ideal. But as is pointed at the beginning, it is important to note that the period of the two continued fractions have no control on each other. As the regulator is concerned, it is better to express the zeta values using the terms of the positive continued fraction. Nevertheless, our earlier work has been made via translation of

the terms of the positive continued fraction into those of negative continued fraction. Hence the direct use of the positive continued fraction significantly reduces the amount of the computations needed. While the negative continued fraction yields an actual cone decomposition, the positive continued fraction gives rise to a virtual cone decomposition. In fact, the cone decomposition appeared at first to be a fan in toric geometry where no virtual decomposition is allowed to define a toric variety. The Todd additivity can be simply extended to virtual decompositions if we take care of the orientation of the cone and take it as the sign of the Todd series. Second, we make a direct use of the Euler-Maclaurin formula of Karshon-Sternberg-Weitsman. Since the exponentials in our case is of Schwarz class toward the infinity of the cone to evaluate. Thus both sides of the Euler-Maclaurin formula make sense and we have more transparent proof. In addition, Karshon-Sternberg-Weitsman's version of Euler-Maclaurin has advantage over Brion-Vergne's in that no limitation on the function to integrate while Brion-Vergne's, since the former can be applied to wider range of functions.

Again, some part of the computation is similar to what had been done by Zagier([34]). He obtained the partial zeta values at non-positive integers again by decomposition of the underlying cone of the zeta summation according to the negative continued fraction together with the Euler-Maclaurin formula. One should note that this decomposition appears in the dual side while the additive decomposition of the Todd differential operator is taken in this paper and [12]. It was already pointed out in [12] as "M-additivity" to "N-additivity". One could ask direct application of Zagier's method with the cone decomposition from the positive continued fraction. Unfortunately, in this setting, the both sides of Euler-Maclaurin formula don't make sense but need certain renormalization process to avoid operations with infinity. In our setting, this is no problem as the domain of integration remains the same while the cone decomposition is reflected to the Todd differential operator.

The plan of this paper is as follows: First, we rewrite the partial zeta function of an ideal as a zeta function of a quadratic form weighted by a fundamental lattice cone of the Shintani decomposition(Sec. 2). We recall a standard asymptotic method to evaluate the zeta values at nonpositive integers and rebuild a version of Euler-Maclaurin formula (Sec.3-4). Then we apply this Euler-Maclaurin formula to obtain an expression of the zeta values and another expression after the cone decomposition arising from the positive continued fractions(Sec.5-7). The partial zeta values at $s = 0, -1, -2$ are explicitly computed using our method and compared with previously known results(Sec.8). Sec.9, which is technical and similar to the computation by Zagier([34]), is devoted to the

proof of vanishing of a part skipped in the previous sections. Finally, we apply this to some families of real quadratic fields to prove our main theorem(Thm.1.1) and the polynomials are explicitly computed out for some families and for some small k (Sec.10).

2. PARTIAL ZETA FUNCTION OF REAL QUADRATIC FIELDS

2.1. Partial zeta function. Let K be a real quadratic field and \mathfrak{b} be an ideal. Through out this article, by partial zeta function, we mean the partial zeta function of an ideal class in narrow sense. The partial zeta function of an ideal \mathfrak{b} is defined as

$$\zeta(s, \mathfrak{b}) := \sum_{\substack{\mathfrak{a} \sim \mathfrak{b} \\ \mathfrak{a}: \text{integral}}} N(\mathfrak{a})^{-s}$$

where $\mathfrak{a} \sim \mathfrak{b}$ means $\mathfrak{b} = \alpha \mathfrak{a}$ for totally positive α in K . This infinite series defines a holomorphic function in the region $\text{Re}(s) > 1$ of the complex plane and has a meromorphic continuation to the entire complex plane. Since for an integral ideal \mathfrak{a} in the narrow class of \mathfrak{b} there exists totally positive element $a \in \mathfrak{b}^{-1}$ such that $\mathfrak{a} = a\mathfrak{b}$ and vice versa, we can write again

$$\zeta(s, \mathfrak{b}) = \sum_{[a] \in (\mathfrak{b}^{-1})^+ / E^+} N(a\mathfrak{b})^{-s},$$

where $(\mathfrak{b}^{-1})^+$ denotes the set of totally positive elements of \mathfrak{b}^{-1} and $E^+ = E_K^+$ denotes the group of totally positive units of K . Now we are going to describe the summation as taken inside the Minkowski space of K . Let (ι_1, ι_2) be two real embeddings of K . Let us denote the Minkowski space of K by

$$K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R} = K_{\iota_1} \times K_{\iota_2}$$

Then one can identify an ideal \mathfrak{c} with a lattice of $K_{\mathbb{R}}$ given by its image under the diagonal embedding of K into $K_{\mathbb{R}}$:

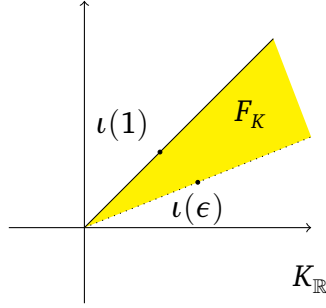
$$\iota = (\iota_1, \iota_2) : K \rightarrow K_{\mathbb{R}}, \quad (\iota(a) \mapsto (\iota_1(a), \iota_2(a)))$$

This is a full lattice in the Minkowski space.

E_K^+ acts on the 1st quadrant of $K_{\mathbb{R}}$ by coordinate-wise multiplication after the diagonal embedding. Let ϵ be the totally positive fundamental unit of K . A fundamental domain of this action is given as a half-open cone F_K of $K_{\mathbb{R}}$ with basis $\{\iota(1), \iota(\epsilon)\}$:

$$F_K = \{x\iota(1) + y\iota(\epsilon) \in \mathbb{R}^2 \mid x \geq 0, y > 0\}.$$

For an ideal \mathfrak{a} of K or a lattice Λ of $K_{\mathbb{R}}$, we denote its intersection with F_K by $F_K(\mathfrak{a})$ or $F_K(\Lambda)$, respectively.

FIGURE 1. A fundamental cone of E_+ -action

For $[a] \in (\mathfrak{b}^{-1})^+/E^+$, there is a unique representative a chosen in $F_K(\mathfrak{b}^{-1})$. Thus we have

$$\zeta(s, \mathfrak{b}) = \sum_{a \in F_K(\mathfrak{b}^{-1})} N(a\mathfrak{b})^{-s}.$$

2.2. Zeta function of 2-dimensional cones. Consider the standard lattice $M := \mathbb{Z}^2$ in \mathbb{R}^2 . Let $Q(x, y) = ax^2 + bxy + cy^2$ be a quadratic form. For two linearly independent vectors v_1, v_2 , let $\sigma(v_1, v_2)$ be the cone in \mathbb{R}^2 as the convex hull of the two rays $\mathbb{R}^+v_1, \mathbb{R}^+v_2$:

$$\sigma(v_1, v_2) := \{x_1v_1 + x_2v_2 \mid x_i > 0 \text{ for } i = 1, 2\}.$$

For simplicity, we write σ instead of $\sigma(v_1, v_2)$ if v_1, v_2 is clear from the context. Following our convention on cones, the origin is not contained in σ .

Define a weight function wt_σ with respect to σ as follows:

$$(2.1) \quad wt_\sigma(\ell) = \begin{cases} 1 & \ell \in \text{int}(\sigma) \\ \frac{1}{2} & \ell \in \partial(\sigma) - (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

This strange weight is justified via identification of the partial zeta function with the zeta function of a lattice cone that will be defined soon below. The partial zeta function is a sum over the points of $F_K(\mathfrak{b}^{-1})$. Since the two edges of F_K are related by the multiplication of the totally positive unit, the summands over both edges coincide. When we take F_K as half-open cone, this repetition is automatically removed. Equivalently, we may apply this weight function so that the total contribution over an orbit equals 1. The choice of assigning 1/2 to each edge will be found useful when we apply the Euler-Maclaurin formula to the cone.

For σ and a quadratic form $Q(-)$ satisfying $Q(v) > 0$ for $v \in \sigma$, we define a zeta function as the following series on $\operatorname{Re}(s) > 1$:

$$\zeta_Q(s, \sigma) := \sum_{m \in M} \frac{wt_\sigma(m)}{Q(m)^s}.$$

2.3. Comparison of zeta functions. One can choose \mathfrak{b} as an integral ideal in the same class such that $\mathfrak{b}^{-1} = [1, \omega]$ for $[1, \omega]$ being the free \mathbb{Z} -module generated by 1 and ω . Taking $\iota(1), \iota(\omega)$ as basis of $K_{\mathbb{R}}$, we have trivialization

$$K_{\mathbb{R}} \simeq \mathbb{R}^2 \quad \text{and} \quad \iota(\mathfrak{b}^{-1}) \simeq M = \mathbb{Z}^2.$$

Here, we fix the order of the basis such that $x + y\omega$ reads (y, x) in \mathbb{R}^2 .

From the reduction theory of quadratic forms, we have a privileged choice of ω such that

$$\iota_1(\omega) > 1, -1 < \iota_2(\omega) < 0.$$

Then the totally positive fundamental unit ϵ belongs to \mathfrak{b}^{-1} and $\epsilon = p + q\omega$ for a pair (p, q) of relatively prime positive integers.

Let σ be lattice cone generated by $(0, 1)$ and (q, p) , which corresponds to F_K . One should be aware that this identification depends on \mathfrak{b} and the choice of ω . Then we have for an integral ideal $\mathfrak{a} = a\mathfrak{b}$ with a totally positive in K

$$N(\mathfrak{a}) = N(m\omega + n)N(\mathfrak{b}) = Q(m, n),$$

for $a = m\omega + n \in F_K(\mathfrak{b}^{-1})$. Thus we have the following identification of zeta functions:

Lemma 2.1. *Let $Q(m, n) = N(m\omega + n)N(\mathfrak{b})$ and σ be a cone defined as above. Then we have*

$$\zeta(s, \mathfrak{b}) = \zeta_Q(s, \sigma).$$

3. EULER-MACLAURIN FORMULA AND ZAGIER'S ASYMPTOTICS

In the previous section, we have identified the partial zeta function of an ideal as a zeta function of a quadratic form defined over a lattice cone.

To evaluate the values at nonpositive integers, we will apply Zagier's asymptotic method to the exponential series associated to the zeta function of quadratic form running over the lattice points of the considered cone. The coefficients of the asymptotic expansion of the exponential series which will be obtained via Euler-Maclaurin formula are the zeta values at nonpositive integers up to some simple factors.

We first recall the asymptotic method of Zagier then state the appropriate Euler-Maclaurin formula for our case.

3.1. Zagier's asymptotic method. For a Dirichelet series of the following form

$$\zeta_{\Delta}(s) := \sum_{\lambda} \frac{a_{\lambda}}{\lambda^s}, \quad \{\lambda\} \subset \mathbb{R}^+, \quad \lambda \rightarrow \infty,$$

if meromorphically continued to the entire complex plane, there is a fairly standard approach to evaluate the values at nonpositive integers.

If $\sum_{\lambda} a_{\lambda} e^{-\lambda t}$ has asymptotic expansion $\sum_{i=-1}^{\infty} c_i t^i$ at $t = 0$, then $\zeta_{\Delta}(s)$ has meromorphic continuation to entire complex plane and

$$\zeta_{\Delta}(-n) = (-1)^n n! c_n$$

for a nonnegative integer n (See Prop.2 in [34]).

Let $Q(-)$ be a quadratic form and σ be a lattice cone such that $Q|_{\sigma}$ is positive. The Dirichlet series defining the zeta function of (Q, σ) yields the following exponential sum

$$(3.1) \quad E(t, Q, \sigma) := \sum_{\ell \in \sigma \cap M} wt_{\sigma}(\ell) e^{-Q(\ell)t},$$

where $wt(-)$ is the weight function defined for σ in Sec. 2.

The asymptotic expansion of the above exponential series will be computed after we state the appropriate version of Euler-Maclaurin formula with remainder for σ and the weight in consideration.

4. EULER-MACLAURIN FORMULA FOR 2-D CONES

4.1. Twisted Todd and L-series. Let λ be an N -th root of 1. We define a λ -twist of the classical Todd series.

$$\text{Todd}^{\lambda}(S) = \frac{S}{1 - \lambda e^{-S}}$$

When $\lambda = 1$, this is the classical Todd series. This version of Todd series is used in [4] where they take the sum of the values of a function at the lattice points (strictly) inside of a simple lattice polytope.

As we use weight $1/2$ on the boundary rays of a cone, another variant of the Todd series with λ -twist is defined as follows: for λ a root of unity,

$$L^{\lambda}(S) = \frac{S}{2} \frac{1 + \lambda e^{-\lambda}}{1 - \lambda e^{-\lambda}} = \frac{S}{1 - \lambda e^{-S}} - \frac{S}{2}.$$

We call this the λ -twisted L -series. This fits well to the case when we put the weight $(1/2)^{\text{cod}}$ on a face, where 'cod' denotes the codimension of the face. This is used in [22] in their version of Euler-Maclaurin formula. For $\lambda = 1$, $L^1(S)$ is nothing but the even part of $\text{Todd}(S)$. Similarly to

the case of Todd series, the series expansion at $S = 0$ of $L^\lambda(S)$ is given as

$$L^\lambda(S) = \left(\frac{1}{2} - \frac{\lambda}{1-\lambda}\right)S + Q_{2,\lambda}(0)S^2 + Q_{3,\lambda}(0)S^3 + \cdots + Q_{k,\lambda}(0)S^k + \cdots,$$

where $Q_{i,\lambda}(x)$ is a (generalized) function of period N for $i = 0, 1, 2, \dots$ defined as follows.

For $m = 0$,

$$(4.1) \quad Q_{0,\lambda}(x) := - \sum_{n \in \mathbb{Z}} \lambda^n \delta(x - n).$$

For $m > 1$, $Q_{m,\lambda}(x)$ is an indefinite integral of $Q_{m-1,\lambda}(x)$ with an integral constant fixed by the boundary value condition

$$(4.2) \quad \int_0^N Q_{m,\lambda}(x) dx = Q_{m+1,\lambda}(N) - Q_{m+1,\lambda}(0) = 0.$$

Thus we have

$$\frac{d}{dx} Q_{m,\lambda}(x) = Q_{m-1,\lambda}(x) \quad \text{and} \quad \int_0^N Q_{m,\lambda}(x) dx = 0.$$

Note that we are taking these $Q_{m,\lambda}$ for $m \geq 0$ as distributions on $L_c^1([0, +\infty))$. $Q_{1,\lambda}$ is continuous and $Q_{m,\lambda}$ is C^{m-1} -function for $m \geq 2$. These generalize the periodic Bernoulli functions appearing in some literatures on analytic continuation of the Riemann zeta function using the Euler-Maclaurin formula(eg. [8]).

4.2. Todd series of 2-dimensional cone. Let $M = \mathbb{Z}^2 \subset \mathbb{R}^2$ be a fixed lattice. Recall that a *lattice cone* is the convex hull of two rays generated by lattice vector. We may assume the generating vectors of a cone are primitive(i.e. not a multiple of other lattice vector in the same ray). For two linearly independent primitive lattice vectors v_1, v_2 , let $\sigma(v_1, v_2)$ be the cone generated by v_1 and v_2 . When v_1, v_2 are clear from the context, we will simply write σ instead of $\sigma(v_1, v_2)$. When there appear several cones, they will be denoted by $\sigma, \tau \dots$ or $\sigma_1, \sigma_2, \sigma_3, \dots$. Since we will be concerned with surface integral over a 2-dimensional cone the order of the basis vectors (ie. the orientation of the cone) is important. So $\sigma(v_1, v_2)$ is never equal to $\sigma(v_2, v_1)$. Taking v_1, v_2 as column vectors in \mathbb{Z}^2 , we associate a nonsingular (2×2) -matrix

$$A_\sigma = (v_1, v_2)$$

to a lattice cone $\sigma = \sigma(v_1, v_2)$. Conversely, if a (2×2) -nonsingular matrix A with integer coefficient has column vectors v_1, v_2 which are primitive, we can associate a unique lattice cone. A cone is said to be

nonsingular if the matrix is in $GL_2(\mathbb{Z})$. Equivalently, σ is nonsingular iff $\det(A_\sigma) = \pm 1$.

Remark 4.1. *In literatures on polytopes or toric geometry, a cone is said to be simple if it is generated by n -linearly independent rays in \mathbb{R}^n . In this article, as we are considering only 2 dimensional cones, every cone is simple unless degenerate.*

Let M_σ be the sublattice of M generated by v_1, v_2 and Γ_σ be M/M_σ . An element $g \in \Gamma_\sigma$ can be written as

$$g = a_{\sigma,1}(g)v_1 + a_{\sigma,2}(g)v_2$$

for rational numbers $a_{\sigma,1}(g), a_{\sigma,2}(g)$ modulo \mathbb{Z} . This is given ambiguously but yields two well-defined characters

$$\chi_{\sigma,i} : g \mapsto e^{2\pi i a_{\sigma,i}(g)}, \quad \text{for } i = 1, 2.$$

The Todd power series for a cone σ is defined as

$$\text{Todd}_\sigma(x_1, x_2) := \sum_{g \in \Gamma_\sigma} \text{Todd}^{\chi_{\sigma,1}(g)}(x_1) \text{Todd}^{\chi_{\sigma,2}(g)}(x_2).$$

Similarly, we define the L-series for σ as

$$L_\sigma(x_1, x_2) := \sum_{g \in \Gamma_\sigma} L^{\chi_{\sigma,1}(g)}(x_1) L^{\chi_{\sigma,2}(g)}(x_2).$$

These are used in the Euler-Maclaurin formula of [4] and [22], respectively. For a 2-dim cone σ , the Todd and the L-series are related in the following manner:

Lemma 4.2.

$$L_\sigma(x_1, x_2) = \text{Todd}_\sigma(x_1, x_2)^{\text{even}} - \frac{|\Gamma_\sigma|}{4} x_1 x_2,$$

where $\text{Todd}_\sigma(x_1, x_2)^{\text{even}}$ denotes the even part of $\text{Todd}_\sigma(x_1, x_2)$ under $(x_1, x_2) \mapsto (-x_1, -x_2)$.

Proof. From the definition of $L^\lambda(S)$, we have

$$L^\lambda(S) = L^{\lambda^{-1}}(-S).$$

Let $\lambda_{\gamma,i} := e^{2\pi i \langle \alpha_i, \gamma \rangle} = \chi_{\sigma,i}(\gamma)$ for $i = 1, 2$. Then we have

$$\begin{aligned} 2\text{Todd}_\sigma^{\text{even}}(x_1, x_2) &= \text{Todd}_\sigma(x_1, x_2) + \text{Todd}_\sigma(-x_1, -x_2) \\ &= \sum_{\gamma \in \Gamma_\sigma} \prod_{i=1}^2 \left(L^{\lambda_{\gamma,i}}(x_i) + \frac{x_i}{2} \right) + \sum_{\gamma \in \Gamma_\sigma} \prod_{i=1}^2 \left(L^{\lambda_{\gamma,i}}(-x_i) - \frac{x_i}{2} \right) \\ &= 2L_\sigma(x_1, x_2) + \frac{|\Gamma_\sigma|}{2} x_1 x_2 \end{aligned}$$

This finishes the proof. □

We say two cones σ_1 and σ_2 are *similar* if

$$AA_{\sigma_1} = A_{\sigma_2}$$

for $A \in GL_2(\mathbb{Z})$. In this case, A induces an isomorphism of M_{σ_1} in M_{σ_2} , which descends to isomorphism of Γ_{σ_1} in Γ_{σ_2} . Since this isomorphism takes the lattice generators of σ_1 to those of σ_2 , the two characters are preserved. *A priori* the Todd series of two cones coincide.

Proposition 4.3. *For two similar cones σ and τ , we have*

$$\text{Todd}_\sigma(x_1, x_2) = \text{Todd}_\tau(x_1, x_2).$$

Proof. Clear. □

One should be aware that this is not the similarity of the matrices in linear algebra. $\sigma(v_1, v_2)$ and $\sigma(v_2, v_1)$ are not similar in general.

4.3. Dual cone and its lattice. Let $N := \text{Hom}(M, \mathbb{Z})$ be the dual lattice of M . N is a lattice in the vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Using the standard inner product $\langle x, y \rangle$ we will often identify N and M . Associated to a lattice cone σ in M , its dual cone $\check{\sigma}$ is defined as

$$\check{\sigma} := \{y \in N_{\mathbb{R}} - 0 \mid \langle y, x \rangle \geq 0\}$$

As the orientation is concerned, $\check{\sigma}$ is endowed with the orientation given by transpose of the matrix of σ . Notice that $\check{\sigma}$ is again a lattice cone generated by two primitive lattice vectors inward and normal to σ . To $\check{\sigma}$, there are two lattices naturally associated. $M_{\check{\sigma}}$ is the sublattice of $M = N$ generated by the primitive lattice vectors of $\check{\sigma}$. Note that this coincides with the definition of M_σ in 4.2. $N_{\check{\sigma}} := \text{Hom}(M_{\check{\sigma}}, \mathbb{Z})$ is a lattice in $N_{\mathbb{R}}$ generated by dual vectors α_1, α_2 to v_1, v_2 if $\sigma = \sigma(v_1, v_2)$ (ie. $\langle \alpha_i, v_j \rangle = \delta_{ij}$). These are related by the following inclusion relation:

$$M_{\check{\sigma}} \subset M = N = \mathbb{Z}^2 \subset N_{\check{\sigma}}$$

4.4. Euler-Maclaurin formula for 2-d cones. Let $L^{k,\lambda}(S)$ be the truncation of $L^\lambda(S)$ below degree k :

$$L^{k,\lambda}(S) = \left(\frac{1}{2} + \frac{\lambda}{1-\lambda}\right)S + Q_{2,\lambda}(0)S^2 + \cdots + Q_{k,\lambda}(0)S^k.$$

For a cone σ , define a weight function as

$$wt_{\sigma}^{KSW}(x) = \begin{cases} 1 & x \in \text{int}(\sigma), \\ 1/2 & x \in \partial\sigma - 0, \\ 1/4 & x = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then the Euler-MacLaurin formula (with remainder) *à la* Karshon-Sternberg-Weitsman in [22] applied to $f(x)$ evaluates the summation of the values of $f(x)$ on the lattice points of σ w.r.t. wt_{σ}^{KSW} .

Theorem 4.4 (Karshon-Sternberg-Weitsman[22]). *Let σ be a lattice cone such that*

$$\check{\sigma} = \check{\sigma}(u_1, u_2) \text{ for } u_i \in \mathbb{Z}^2.$$

Let α_i be a dual basis of u_i (ie. $\langle \alpha_i, u_j \rangle = \delta_{ij}$) and $\lambda_{\gamma,i} := e^{2\pi i \langle \gamma, \alpha_i \rangle}$ for $i, j = 1, 2$. Let

$$\sigma(h) = \{x \in \mathbb{R}^2 \mid \langle x, u_i \rangle \geq -h_i, \text{ for } i = 1, 2\}$$

for $h = (h_1, h_2) \in \mathbb{R}^2$. Suppose f is a smooth integrable function on $\sigma(h)$ for some $h \in (\mathbb{R}_+)^2$. Assume further that f is rapidly decaying toward the infinity of σ . Then we have

$$\sum_{x \in \sigma \cap \mathbb{Z}^2} wt_{\sigma}^{KSW}(x) f(x) = \sum_{\gamma \in \Gamma_{\check{\sigma}}} \left(\prod_{i=1}^2 L^{k, \lambda_{\gamma,i}}(\partial_{h_i}) \right) \int_{\sigma(h)} f(x) dx \Big|_{h=0} + R_k^{\sigma}(f).$$

where

$$\begin{aligned} R_k^{\sigma}(f) = & \frac{1}{|\Gamma_{\check{\sigma}}|} \sum_{\gamma \in \Gamma_{\check{\sigma}}} \int_0^{\infty} \int_0^{\infty} Q_{k, \lambda_{\gamma,2}}(x_2) \partial_{x_2}^k Q_{k, \lambda_{\gamma,1}}(x_1) \partial_{x_1}^k f(x_1 \alpha_1 + x_2 \alpha_2) dx_1 dx_2 \\ & + \frac{1}{|\Gamma_{\check{\sigma}}|} \sum_{\gamma \in \Gamma_{\check{\sigma}}} (-1)^{k-1} \int_0^{\infty} \int_0^{\infty} Q_{k, \lambda_{\gamma,2}}(x_2) \partial_{x_2}^k L^{k, \lambda_{\gamma,1}}(-\partial_{x_1}) f(x_1 \alpha_1 + x_2 \alpha_2) dx_1 dx_2 \\ & + \frac{1}{|\Gamma_{\check{\sigma}}|} \sum_{\gamma \in \Gamma_{\check{\sigma}}} (-1)^{k-1} \int_0^{\infty} \int_0^{\infty} Q_{k, \lambda_{\gamma,1}}(x_1) L^{k, \lambda_{\gamma,2}}(-\partial_{x_2}) \partial_{x_1}^k f(x_1 \alpha_1 + x_2 \alpha_2) dx_1 dx_2. \end{aligned}$$

In the above theorem, the summation is weighted with wt_{σ}^{KSW} . Now we come to the same theorem with the weight changed to wt_{σ} in (2.1) for our purpose.

Corollary 4.5. *With the same notations as in Thm.4.4, we have that*

$$\sum_{x \in \sigma \cap \mathbb{Z}^2} wt_{\sigma}(x) f(x) = \left[\sum_{\gamma \in \Gamma_{\check{\sigma}}} \left(\prod_{i=1}^2 L^{k, \lambda_{\gamma,i}}(\partial_{h_i}) - \frac{1}{4} \partial_{h_1} \partial_{h_2} \right) \right] \int_{\sigma(h)} f(x) dx \Big|_{h=0} + R_k^{\sigma}(f).$$

Proof. This comes from the following observation:

$$\begin{aligned} f(0,0) &= \partial_{h_1} \partial_{h_2} \int_{-h_2}^{\infty} \int_{-h_1}^{\infty} f(\alpha_1 x_1 + \alpha_2 x_2) dx_1 dx_2 \Big|_{h=0} \\ &= |\Gamma_{\check{\sigma}}| \partial_{h_1} \partial_{h_2} \int_{\sigma(h)} f(y) dy \Big|_{h=0}. \end{aligned}$$

and

$$\sum_{x \in \sigma \cap \mathbb{Z}^2} wt_{\sigma}(x) f(x) = \sum_{x \in \sigma \cap \mathbb{Z}^2} wt_{\sigma}^{KSW}(x) f(x) - \frac{1}{4} f(0,0).$$

□

4.5. Asymptotic expansion. For $\lambda = e^{2\pi i \frac{j}{N}}$ an N -th root of 1 and $k \geq 2$, from the periodicity one sees that $Q_{k,\lambda}(x)$ is a bounded continuous function. Since we are assuming that f decays rapidly toward $x, y \rightarrow +\infty$, we have

$$\int_0^{\infty} \int_0^{\infty} \partial_{x_1}^i \partial_{x_2}^j f(\alpha_1 x_1 + \alpha_2 x_2) dx_1 dx_2$$

is bounded for any $i, j \geq 0$.

Thus the summation of the values of $f(tx)$ over x running over lattice points of σ is expressed in series occurring in Euler-MacLaurin formula involving the L -series.

Theorem 4.6. *Under the same assumption on σ as in Thm.4.4, we have the asymptotic expansion for $t \in \mathbb{R}^+$ and $t \rightarrow 0$*

$$\sum_{x \in \sigma \cap \mathbb{Z}^2} wt_{\sigma}(x) f(tx) \sim \left(L_{\check{\sigma}}(\partial_{h_1}, \partial_{h_2}) - \frac{|\Gamma_{\sigma}|}{4} \partial_{h_1} \partial_{h_2} \right) \circ \int_{\sigma(h)} f(tx) dx \Big|_{h=0}.$$

Proof. We note that

$$\int_{\sigma(h)} f(tx) dx = t^{-2} \int_{\sigma(th)} f(x) dx.$$

Applying Thm. 4.4, for $t \in \mathbb{R}^+$, we have

$$\sum_{x \in \sigma \cap \mathbb{Z}^2} wt_{\sigma}(x) f(tx) = \sum_{\gamma \in \Gamma_{\check{\sigma}}} \left(t^{-2} \prod_{i=1}^2 L^{k, \lambda_{\gamma}, i} \left(t \frac{\partial}{\partial h_i} \right) - \frac{1}{4} \partial_{h_1} \partial_{h_2} \right) \int_{\sigma(h)} f(x) dx \Big|_{h=0} + R_k^{\sigma}(f)(t),$$

with

$$\begin{aligned}
& R_k^\sigma(f)(t) \\
&= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_0^\infty \int_0^\infty Q_{k,\lambda_{\gamma,2}}(x_2) Q_{k,\lambda_{\gamma,1}}(x_1) \partial_{x_2}^k \partial_{x_1}^k f(\alpha_1 t x_1 + \alpha_2 t x_2) dx_1 dx_2 \\
&\quad + \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (-1)^{k-1} \int_0^\infty \int_0^\infty Q_{k,\lambda_{\gamma,2}}(x_2) \partial_{x_2}^k L^{k,\lambda_{\gamma,1}}(-\partial_{x_1}) f(\alpha_1 t x_1 + \alpha_2 t x_2) dx_1 dx_2 \\
&\quad + \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (-1)^{k-1} \int_0^\infty \int_0^\infty Q_{k,\lambda_{\gamma,1}}(x_1) L^{k,\lambda_{\gamma,2}}(-\partial_{x_2}) \partial_{x_1}^k f(\alpha_1 t x_1 + \alpha_2 t x_2) dx_1 dx_2,
\end{aligned}$$

for $\Gamma = \Gamma_{\check{\sigma}}$.

Let us change the variables: $y_i = t x_i$ for $i = 1, 2$. Then from the boundedness of $Q_{k,\lambda_{\gamma,i}}$, as $f(x, y)$ decays rapidly, one can easily see for example a summand of $R_k^\sigma(f)(t)$ in the second line belongs to $O(t^{k-1})$ at 0:

$$\begin{aligned}
& \int_0^\infty \int_0^\infty Q_{k,\lambda_{\gamma,2}}(x_2) \partial_{x_2}^k L^{k,\lambda_{\gamma,1}}(-\partial_{x_1}) f(\alpha_1 t x_1 + \alpha_2 t x_2) dx_1 dx_2 \\
&= t^{k-2} \int_0^\infty \int_0^\infty Q_{k,\lambda_{\gamma,2}}\left(\frac{y_2}{t}\right) \partial_{y_2}^k L^{k,\lambda_{\gamma,1}}(-t \partial_{y_1}) f(\alpha_1 y_1 + \alpha_2 y_2) dy_1 dy_2 \\
&= O(t^{k-1})
\end{aligned}$$

Similarly, one can show the rest belongs to $O(t^{k-1})$. Thus we conclude that

$$R_k^\sigma(f)(t) = O(t^{k-1}).$$

□

The asymptotic expansion in theorem 4.6 can be rewritten using todd power series from lemma 4.2.

Theorem 4.7. *For $t \in \mathbb{R}^+$ and $t \rightarrow 0$, we have the asymptotic expansion*

$$\sum_{x \in \sigma \cap \mathbb{Z}^2} w t_\sigma(x) f(tx) \sim \left(\text{Todd}_{\check{\sigma}}^{\text{even}}(\partial_{h_1}, \partial_{h_2}) - \frac{q}{2} \partial_{h_1} \partial_{h_2} \right) \circ \int_{\sigma(h)} f(tx) dx \Big|_{h=0}.$$

4.6. Evaluation of zeta values. Now we apply this to the exponential series associated to the partial zeta function of an ideal. We saw that a partial zeta function can be identified with a zeta function of a cone σ w.r.t. a quadratic form Q . We are going to apply Zagier's theorem to evaluate the special values of $\zeta(\mathfrak{b}, s)$ at non positive integers. We apply Thm. 4.7 to obtain the asymptotic expansion of the exponential series. We take $f(tx) = e^{-Q(t^{1/2}x_1, t^{1/2}x_2)} = e^{-Q(x_1, x_2)t}$. Then we obtain the

following asymptotic expansion:

$$\sum_{l \in \sigma \cap M} w t_{\sigma} e^{-Q(l)t} \sim \left\{ (\text{Todd}_{\check{\sigma}}^{\text{even}}(\partial_{h_1}, \partial_{h_2}) - \frac{q}{2} \partial_{h_1} \partial_{h_2}) \circ \int_{\sigma(h_1, h_2)} e^{-Q(x_1, x_2)t} dx_1 dx_2 \right\} \Big|_{(h_1, h_2)=0}$$

Theorem 4.8 (Garoufalidis-Pommersheim[12]). *For $n \geq 0$, we have*

$$\zeta(\mathfrak{b}, -n) = (-1)^n n! \left\{ \text{Todd}_{\check{\sigma}}^{(2n+2)}(\partial_{h_1}, \partial_{h_2}) - \delta_{n,0} \frac{q}{2} \partial_{h_1} \partial_{h_2} \right\} \circ \int_{\sigma(h_1, h_2)} e^{-Q(x_1, x_2)t} dx_1 dx_2 \Big|_{(h_1, h_2)=0}$$

where

$$\delta_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.9. In [12], they used the exact Euler-Maclaurin formula of Brion-Vergne([4]) to a cone over a polytope. We apply the version with remainder term due to Karshon-Sternberg-Weitsman in [22]. Since this is a formula with remainder, we have a direct method to evaluate the zeta values at nonpositive integers.

5. ADDITIVITY OF TODD SERIES AND CONE DECOMPOSITION

In this section, we recall some technic used in [12] concerning additive decomposition of the Todd series w.r.t. cone decomposition.

Todd series does not allow decomposition in its original shape until we normalize. The *normalized todd power series* S_{σ} for a cone σ is defined as follows:

$$S_{\sigma}(x_1, x_2) = \frac{1}{\det(A_{\sigma})x_1x_2} \text{Todd}_{\sigma}(x_1, x_2).$$

One should note that different choice of the orientation of the same underlying cone yields the opposite sign in the normalized Todd series and interchanges the two variables. This is contrary to the original Todd series case, where the similarity class is determined by the sign.

Let $v_i \in \mathbb{R}^2$ for $i = 1, 2, 3$ be pairwise linearly independent primitive lattice vectors in a half-plane. An ordered pair (v_i, v_j) for $i \neq j$ determines a lattice cone $\sigma_{ij} = \sigma_{ij}(v_i, v_j)$ with orientation.

In this case, we write formally

$$\sigma_{ij} + \sigma_{jk} = \sigma_{ik}.$$

Then we have the following:

Theorem 5.1 (Garoufalidis-Pommersheim [12]). *For $i = 1, \dots, r+1$, let v_i be pairwise linearly independent lattice points in a half plane of \mathbb{R}^2 . We define cones*

$$\sigma_i := \sigma_i(v_i, v_{i+1}), \quad \sigma := \sigma(v_1, v_{r+1})$$

Thus

$$\sigma = \sigma_1 + \sigma_2 + \dots + \sigma_r.$$

Then

$$S_\sigma(x_1, x_2) = \sum_{i=1}^r S_{\sigma_i}(A_{\sigma_i}^{-1} A_\sigma(x_1, x_2)^t).$$

In particular, if every σ_i is nonsingular (i.e. $\det(A_{\sigma_i}) = \pm 1$) for $i = 1, 2, \dots, r$,

$$S_\sigma(x_1, x_2) = \sum_{i=1}^r \det(A_{\sigma_i}) F(A_{\sigma_i}^{-1} A_\sigma(x_1, x_2)^t),$$

where $F(x_1, x_2) = \frac{1}{1-e^{-x_1}} \frac{1}{1-e^{-x_2}}$.

Proof. See Thm. 2 in [28]. \square

Remark 5.2. Abusing the notation, we denote $\sigma(v_2, v_1)$ by $-\sigma(v_1, v_2)$. Actually by definition of Todd power series of cone above, we easily find that

$$\text{Todd}_\sigma(x_1, x_2) = \text{Todd}_{-\sigma}(x_2, x_1).$$

The matrix A_σ^{-1} represents the linear transformation $v_1 \mapsto e_1, v_2 \mapsto e_2$.

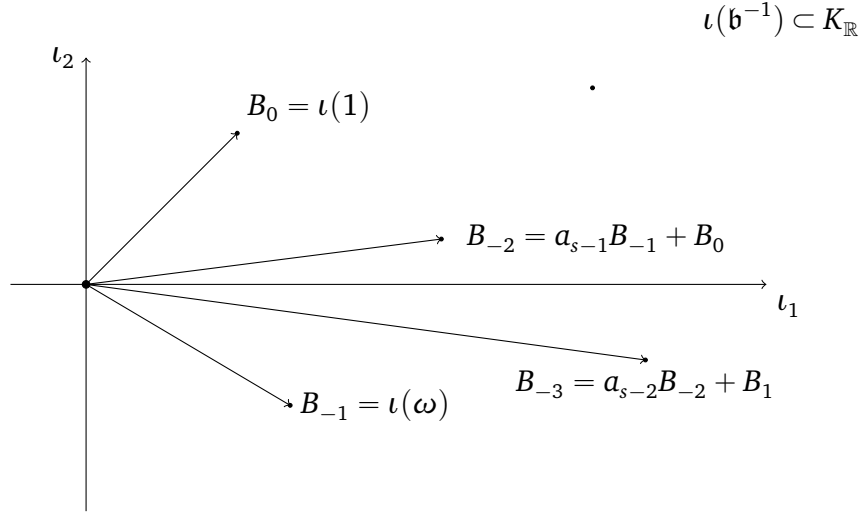
So we have $A_{-\sigma}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_\sigma^{-1}$. Let $A_\sigma^{-1} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ for two row vectors w_1, w_2 . Then $A_{-\sigma}^{-1} = \begin{pmatrix} w_2 \\ w_1 \end{pmatrix}$. Therefore,

$$\begin{aligned} \text{Todd}_\sigma(A_\sigma^{-1}(x_1, x_2)^t) &= \text{Todd}_\sigma(\langle w_1, (x_1, x_2) \rangle, \langle w_2, (x_1, x_2) \rangle) \\ (5.1) \quad &= \text{Todd}_{-\sigma}(\langle w_2, (x_1, x_2) \rangle, \langle w_1, (x_1, x_2) \rangle) \\ &= \text{Todd}_{-\sigma}(A_{-\sigma}^{-1}(x_1, x_2)^t). \end{aligned}$$

Thus one can see easily that for the additivity theorem to hold the orientation of σ does not make any problem.

6. CONE DECOMPOSITION AND CONTINUED FRACTION

In this section, we will decompose the cone $\sigma(\mathfrak{b}^{-1})$ into nonsingular cones. This decomposition follows directly the decomposition of the fundamental cone in the totally positive quadrant of Minkowski space under the action of the totally positive unit group. This is fairly standard fact related to desingularization of a cusp of the Hilbert modular

FIGURE 2. B_i and the continued fraction $[[a_0, a_1, \dots, a_{r-1}]]$

surface of the real quadratic field considered. It is described in terms of the (minus) continued fraction expansion of the reduced basis of \mathfrak{b}^{-1} so that the desingularization of the lattice cone $\sigma(\mathfrak{b}^{-1})$ in the sense of toric geometry follows(cf. [11], [14]). We are going to apply Thm. 5.1 to obtain explicit formula of the zeta values using the terms of the positive continued fraction. One should note that our expression is differed from [12] in that we use the positive continued fraction instead of the negative one.

In general, there are many other decompositions possible for a singular cone. But those lattice cones arising from a singular cone but for a totally real field and the action of the totally positive units has a decomposition after the shape of its Klein polyhedron which is a geometric realization of a continued fraction. In 2 dimension, this appears as follows: In each quadrant of $K_{\mathbb{R}}$, we take the convex hull of $\mathfrak{b}^{-1} = [1, \omega]$ and union the polygonal hulls. This is the *Klein polyhedra* of the ideal lattice \mathfrak{b}^{-1} . One can further assume ω is a reduced basis: $\omega > 1$, $-1 < \omega' < 0$ for $\omega \in K$. Then ω has purely periodic positive continued fraction expansion

$$\omega = [[a_0, a_1, \dots, a_{r-1}]].$$

Let $\{B_{2i}\}$ (resp. $\{B_{2i+1}\}$) be the vertices of the convex hull of $\iota(\mathfrak{b}^{-1})$ in the 1st (resp. the 4th) quadrant of \mathbb{R}^2 with $B_0 = \iota(1), B_{-1} = \iota(\omega)$ and $x(B_i) < x(B_{i-1})$, where $x(-)$ is taking the 1st coordinate. These B_i arising as the vertices of the Klein polyhedron should not be confused with the Bernoulli numbers.

Let ℓ be the even period of the continued fraction expansion of ω (ie. $\ell = r$ (resp. $2r$) for even r (resp. for odd r)).

B_i satisfies a periodic recursive relation read from the continued fraction of ω (cf. [14]):

$$(6.1) \quad B_{i-1} = a_i B_i + B_{i+1}$$

Since a successive pair B_i, B_{i+1} is a basis of the lattice $\iota(\mathfrak{b}^{-1})$ in $K_{\mathbb{R}}$, this yields a change of basis

$$(B_{i-1} \ B_i) = (B_i \ B_{i+1}) \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$$

After successive change of basis, we have

$$(6.2) \quad (B_{i-1} \ B_i) = (B_{-1} \ B_0) \begin{pmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\ell-i} & 1 \\ 1 & 0 \end{pmatrix}$$

Let α_i, β_i be the coordinate of B_{i-1} w.r.t. the basis $\{B_{-1}, B_0\}$:

$$B_{i-1} = \alpha_i B_{-1} + \beta_i B_0$$

Note that the column vector $(\alpha_i, \beta_i)^t$ is equal to the 1st column of the matrix $\begin{pmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\ell-i} & 1 \\ 1 & 0 \end{pmatrix}$ in (6.2).

As B_i are primitive, so is (α_i, β_i) in $M = \mathbb{Z}^2$.

In the following, the totally positive fundamental lemma is identified:

Lemma 6.1. *Let ϵ be the totally positive fundamental unit of K . Then*

$$\iota(\epsilon) = B_{-\ell} = \alpha_{\ell-1} B_{-1} + \beta_{\ell-1} B_0.$$

Proof. Let $\epsilon_K > 1$ be the fundamental unit of K . Then for the period r of continued fraction expansion of ω , we have

$$\iota(\epsilon_K) = B_{-r}.$$

See p.40 of [14] for detail. Since the totally positive unit ϵ is either ϵ_K or ϵ_K^2 according to the sign of $\iota_2(\epsilon_K)$, we then obtain that $\iota(\epsilon) = B_{-\ell}$. \square

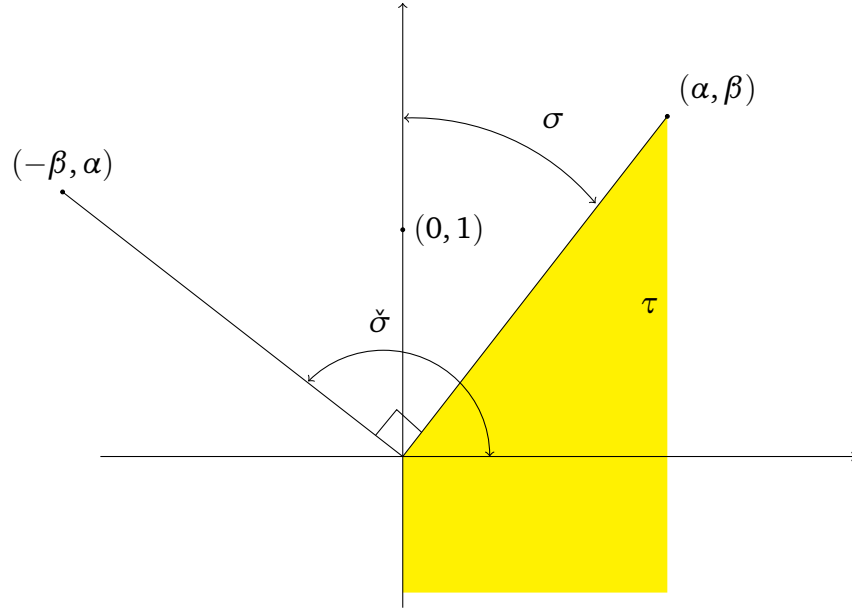
Recall that we associated a lattice cone $\sigma(\mathfrak{b}^{-1})$ in \mathbb{R}^2 to \mathfrak{b}^{-1} in Sec. 2.

$$(6.3) \quad \sigma(\mathfrak{b}^{-1}) := \sigma((0, 1), (\alpha_{\ell-1}, \beta_{\ell-1})).$$

This corresponds to the cone bounded by $\iota(1)$ and $\iota(\epsilon)$ in $K_{\mathbb{R}}$. For the rest of this section, only the cone $\sigma(\mathfrak{b}^{-1})$ is need to consider to compute the zeta values. So we will write simply σ instead of $\sigma(\mathfrak{b}^{-1})$.

Lemma 6.2. *Let $\sigma = \sigma((0, 1), (\alpha, \beta))$ be a lattice cone where α, β are relatively prime positive integers. Then the dual cone of σ is similar to*

$$\tau = \tau((0, -1), (\alpha, \beta)).$$

FIGURE 3. σ , $\check{\sigma}$ and τ

Proof. It is easy to see the dual cone $\check{\sigma}$ has primitive basis $((1,0), (-\beta, \alpha))$. See Fig. 3. Since the rotation by -90 degree belongs to $SL_2(\mathbb{Z})$, we have the desired similarity of the cones. \square

After Prop. 4.3 and Lemma 6.2, for \mathfrak{b} as before, we have

$$\check{\sigma} \sim \tau := \tau((0, -1), (\alpha_{\ell-1}, \beta_{\ell-1}))$$

thus

$$S_{\check{\sigma}}(x_1, x_2) = S_{\tau}(x_1, x_2).$$

Let $v_{-1} = (0, 1)$, $v_0 = (1, 0)$ and for $1 \leq i \leq \ell - 1$,

$$v_i = (\alpha_i, \beta_i),$$

for α_i, β_i defined as in Eq.(6.2). Notice that v_i corresponds to B_{-i+1} and v_{-1}, v_0 are the two standard basis of M . Then the decomposition of σ yields that of $\check{\sigma}$:

Proposition 6.3. *With above notations, let $\sigma'_0 := \sigma'_0(-v_{-1}, v_0)$ and*

$$\sigma_i := \sigma_i(v_{i-1}, v_i),$$

for $i \geq 0$. Then we have

$$\check{\sigma} \sim \tau := \tau(v_{-1}, v_{\ell}) = \sigma'_0 + \sigma_1 + \sigma_2 + \sigma_3 + \cdots + \sigma_{\ell-1}.$$

Thus we have

$$S_{\check{\sigma}}(x_1, x_2) = F(A_{\sigma'_0}^{-1}A_{\tau}(x_1, x_2)^t) + \sum_{i=1}^{\ell-1} (-1)^i F(A_{\sigma_i}^{-1}A_{\tau}(x_1, x_2)^t).$$

Proof. First, one should notice that

$$\tau = \sigma'_0 + \rho$$

where $\rho = \rho(v_0, v_{\ell-1})$ (See Fig. 3). Note also

$$\det(A_{\sigma_i}) = \det \begin{pmatrix} \alpha_{i-1} & \alpha_i \\ \beta_{i-1} & \beta_i \end{pmatrix} = \beta_i \alpha_{i-1} - \alpha_i \beta_{i-1} = (-1)^{i-1}.$$

Hence the decomposition of ρ into nonsingular cones σ_i

$$\rho = \sigma_1 + \sigma_2 + \cdots + \sigma_{\ell-1}$$

finishes the proof. \square

Lemma 6.4. For $-1 \leq i \leq \ell - 1$, let

$$M_i := (-1)^{i+1} ((\beta_i \alpha_{\ell-1} - \alpha_i \beta_{\ell-1}) x_2 + \alpha_i x_1).$$

Then we have

$$\text{Todd}_{\check{\sigma}}(x_1, x_2) = \alpha_{\ell-1} x_1 x_2 \left(\sum_{i=-1}^{\ell-2} (-1)^i F(M_i, M_{i+1}) + \frac{1}{1 - e^{-\alpha_{\ell-1} x_2}} \right),$$

where $F(x_1, x_2) = \frac{1}{1 - e^{-x_1}} \frac{1}{1 - e^{-x_2}}$.

Proof. After simple computation, we obtain that

$$A_{\sigma_{i-1}}^{-1} A_{\tau}(x_1, x_2)^t = (M_{i+1}, M_i)^t.$$

Since $F(-x_1, x_2) + F(x_1, x_2) = \frac{1}{1 - e^{-x_2}}$, we have

$$\begin{aligned} & F(C_0'^{-1} \tau(x_1, x_2)^t) + F(C_0^{-1} \tau(x_1, x_2)^t) \\ &= F(x_1 - \beta_{\ell-1} x_2, \alpha_{\ell-1} x_2) + F(-(x_1 - \beta_{\ell-1} x_2), \alpha_{\ell-1} x_2) \\ &= \frac{1}{1 - e^{-\alpha_{\ell-1} x_2}} \end{aligned}$$

Note

$$\det \begin{pmatrix} 0 & \alpha_{\ell-1} \\ -1 & \beta_{\ell-1} \end{pmatrix} = \alpha_{\ell-1}.$$

If we apply the above to Prop. 6.3, we complete the proof. \square

Let $\text{Todd}_{\sigma}(x_1, x_2)^{(n)}$ be the degree n homogeneous part of $\text{Todd}_{\sigma}(x_1, x_2)$.

Proposition 6.5. *Let*

$$L_k(X, Y) = \sum_{i=1}^{2k+1} \frac{B_i}{i!} \frac{B_{2k+2-i}}{(2k+2-i)!} X^{i-1} Y^{2k-i+1},$$

$$R_k(X, Y) = X^{2k} + X^{2k-1}Y + \dots + Y^{2k}.$$

Then we have

$$\begin{aligned} \text{Todd}_{\delta}(x_1, x_2)^{(2k+2)} = & \alpha_{\ell-1} \left(\sum_{i=-1}^{\ell-2} (-1)^i L_k(M_{i+1}, M_i) x_1 x_2 + \sum_{i=1}^{\ell-1} (-1)^i a_{\ell-i} R_k(M_{i-2}, M_i) x_1 x_2 \frac{B_{2k+2}}{(2k+2)!} \right) \\ & + \frac{B_{2k+2}}{(2k+2)!} (-x_1 M_0^{2k+1} + x_2 M_{\ell-2}^{2k+1}) + \delta_{k,0} \frac{1}{2} \alpha_{\ell-1} x_1 x_2. \end{aligned}$$

Proof. From Prop. 6.4, we find that

$$\text{Todd}_{\delta}(x_1, x_2)^{(2k+2)} = \alpha_{\ell-1} x_1 x_2 \sum_{i=-1}^{\ell-2} (-1)^i F(M_i, M_{i+1})^{(2k)} + x_1 \frac{\alpha_{\ell-1} x_2}{1 - e^{-\alpha_{\ell-1} x_2}}^{(2k+1)}.$$

We have

$$\begin{aligned} & F(M_i, M_{i+1})^{(2k)} \\ &= \sum_{m=1}^{2k+1} \frac{B_m}{m!} \frac{B_{2k+2-m}}{(2k+2-m)!} M_{i+1}^{m-1} M_i^{2k-m+1} + \frac{B_{2k+2}}{(2k+2)!} \left(\frac{M_i^{2k+1}}{M_{i+1}} + \frac{M_{i+1}^{2k+1}}{M_i} \right) \\ &= L_k(M_{i+1}, M_i) + \frac{B_{2k+2}}{(2k+2)!} \left(\frac{M_i^{2k+1}}{M_{i+1}} + \frac{M_{i+1}^{2k+1}}{M_i} \right) \end{aligned}$$

and

$$\frac{\alpha_{\ell-1} x_2}{1 - e^{-\alpha_{\ell-1} x_2}}^{(2k+1)} = -\frac{B_{2k+1}}{(2k+1)!} \alpha_{\ell-1}^{2k+1} x_2^{2k+1} = \delta_{k,0} \frac{1}{2} \alpha_{\ell-1} x_2,$$

as $B_{2k+1} = 0$ for $k > 0$.

Moreover we also have the following:

$$\begin{aligned} (6.4) \quad & \sum_{i=-1}^{\ell-2} (-1)^i (M_{i+1}^{-1} M_i^{2k+1} + M_{i+1}^{2k+1} M_i^{-1}) \\ &= -\frac{M_0^{2k+1}}{M_{-1}} + \frac{M_{\ell-2}^{2k+1}}{M_{\ell-1}} + \sum_{i=1}^{\ell-1} (-1)^i \frac{M_{i-2}^{2k+1} - M_i^{2k+1}}{M_{i-1}}. \end{aligned}$$

As

$$\begin{cases} (\alpha_{-1}, \beta_{-1}) = (0, 1) \\ \alpha_{i+1} = a_{\ell-i-1} \alpha_i + \alpha_{i-1} \\ \beta_{i+1} = a_{\ell-i-1} \beta_i + \beta_{i-1} \end{cases}$$

we have

$$M_{-1} = \beta_{-1}\alpha_{\ell-1}x_2 - \alpha_{-1}(-x_1 + \beta_{\ell-1}x_2) = \alpha_{\ell-1}x_2,$$

$$M_{\ell-1} = \beta_{\ell-1}\alpha_{\ell-1}x_2 - \alpha_{\ell-1}(-x_1 + \beta_{\ell-1}x_2) = \alpha_{\ell-1}x_1.$$

and

$$M_{i+1} = -a_{\ell-i-1}M_i + M_{i-1}.$$

Therefore (6.4) is equal to

$$(6.5) \quad \begin{aligned} & -\frac{M_0^{2k+1}}{\alpha_{\ell-1}x_2} + \frac{M_{\ell-2}^{2k+1}}{\alpha_{\ell-1}x_1} + \sum_{i=1}^{\ell-1} (-1)^i a_{\ell-i} \frac{M_{i-2}^{2k+1} - M_i^{2k+1}}{M_{i-2} - M_i} \\ & = -\frac{M_0^{2k+1}}{\alpha_{\ell-1}x_2} + \frac{M_{\ell-2}^{2k+1}}{\alpha_{\ell-1}x_1} + \sum_{i=1}^{\ell-1} (-1)^i a_{\ell-i} R_k(M_{i-2}, M_i) \end{aligned}$$

Thus we finally complete proof. \square

7. SPECIAL VALUES OF ZETA FUNCTION

Now we are going to evaluate the values of $\zeta(s, \mathfrak{b})$ at non-positive integers using the expression of the degree n homogeneous part of the Todd series made in the previous section. We suppose \mathfrak{b} is an integral ideal normalized as in the previous section so that $\mathfrak{b}^{-1} = [1, \omega]$ for

$$\omega = [[a_0, a_1, \dots, a_{r-1}]].$$

and $\epsilon > 1$ denotes the totally positive fundamental unit of K . Let ℓ be the even period of continued fraction expansion of ω .

(α_i, β_i) for $i = 1, 2, \dots, \ell - 1$, and

$$(\alpha_{-2}, \beta_{-2}) = (1, -a_0), (\alpha_{-1}, \beta_{-1}) = (0, 1), (\alpha_0, \beta_0) = (1, 0)$$

are primitive lattice vectors in M . Note that (α_i, β_i) corresponds to B_{-i+1} in $K_{\mathbb{R}}$.

$$Q(x_1, x_2) := N(\mathfrak{b})(x_1\omega + x_2)(x_1\omega' + x_2).$$

Then the partial zeta function $\zeta(s, \mathfrak{b})$ is expressed as (See Prop.2.1.)

$$\zeta(s, \mathfrak{b}) = \sum_{l \in M} \frac{wt_{\sigma(\mathfrak{b}^{-1})}^2(l)}{Q(l)^s}.$$

where

$$\sigma(\mathfrak{b}^{-1}) = \sigma((0, 1), (\alpha_{\ell-1}, \beta_{\ell-1})).$$

In Thm.4.8., the partial zeta value is written using the Todd differential operator of the cone $\check{\sigma}$ dual to $\sigma = \sigma(\mathfrak{b}^{-1})$. We apply the additivity of the Todd series(Prop.6.5.) after the cone decomposition of $\check{\sigma}$ occurring in the continued fraction of ω to this expression. Then we obtain the following expression of the partial zeta value:

$$(7.1) \quad \zeta(-k, \mathfrak{b}) = (-1)^k k! (\mathcal{L} + \mathcal{R}) \circ \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 \Big|_{h=0}$$

where

$$(7.2) \quad \begin{aligned} \mathcal{L} := & \sum_{i=-1}^{\ell-2} (-1)^i L_k(M_{i+1}, M_i) (\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} \\ & + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i} R_k(M_{i-2}, M_i) (\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} \end{aligned}$$

and

$$(7.3) \quad \begin{aligned} \mathcal{R} := & \frac{B_{2k+2}}{(2k+2)!} \left(-a_{\ell} R_k(M_{-2}, M_0) (\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} - \partial_{h_1} M_0^{2k+1} (\partial_{h_1}, \partial_{h_2}) \right. \\ & \left. + \partial_{h_2} M_{\ell-2}^{2k+1} (\partial_{h_1}, \partial_{h_2}) \right). \end{aligned}$$

In (7.1), as the differential operators are linear, this expression can be evaluated one by one. Later in Sec.9 we will prove that the part of (7.1) involving \mathcal{R} vanishes:

$$(7.4) \quad \mathcal{R} \circ \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 \Big|_{h=0} = 0$$

Then it remains only to evaluate the part involving \mathcal{L} . First, we need to rewrite the integral in another coordinate (y_1, y_2) such that $(x_1, x_2) = (\alpha_{\ell-1} y_2, \beta_{\ell-1} y_2 + y_1)$. So we have $\sigma(h)$ in the new coordinate:

$$\begin{aligned} \sigma(h) &= \sigma(h_1, h_2) \\ &= \{y_1 v_1 + y_2 v_2 \mid (y_1 v_1 + y_2 v_2, u_1) \geq -h_1, (y_1 v_1 + y_2 v_2, u_2) \geq -h_2\} \\ &= \{(\alpha_{\ell-1} y_2, \beta_{\ell-1} y_2 + y_1) \mid x_1 \geq -\frac{h_1}{\alpha_{\ell-1}}, x_2 \geq -\frac{h_2}{\alpha_{\ell-1}}\}. \end{aligned}$$

In the new coordinate (y_1, y_2) the integral becomes

$$(7.5) \quad \begin{aligned} \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 &= \alpha_{\ell-1} \int_{-\frac{h_2}{\alpha_{\ell-1}}}^{\infty} \int_{-\frac{h_1}{\alpha_{\ell-1}}}^{\infty} e^{-Q(\alpha_{\ell-1} y_2, \beta_{\ell-1} y_2 + y_1)} dy_1 dy_2 \\ &= \alpha_{\ell-1} \int_{-\frac{h_2}{\alpha_{\ell-1}}}^{\infty} \int_{-\frac{h_1}{\alpha_{\ell-1}}}^{\infty} e^{-N(\mathfrak{b})N(\epsilon y_2 + y_1)} dy_1 dy_2. \end{aligned}$$

This integral applied by $\alpha_{\ell-1} \partial_{h_1} \partial_{h_2}$ is

$$(7.6) \quad \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 = e^{-N(\mathfrak{b})N(\frac{h_2}{\alpha_{\ell-1}}\epsilon + \frac{h_1}{\alpha_{\ell-1}})}$$

The above simplifies (7.1) quite much assuming the vanishing of (7.4):
(7.7)

$$\begin{aligned} \zeta(-k, \mathfrak{b}) &= (-1)^k k! \left(\sum_{i=-1}^{\ell-2} L_k(M_{i+1}, M_i)(\partial_{h_1}, \partial_{h_2}) \right. \\ &\quad \left. + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i} R_k(M_{i-2}, M_i)(\partial_{h_1}, \partial_{h_2}) \right) \circ e^{-N(\mathfrak{b})N(\frac{h_2}{\alpha_{\ell-1}}\epsilon + \frac{h_1}{\alpha_{\ell-1}})} \Big|_{h=0} \end{aligned}$$

Lemma 7.1. *Let $A_i = \alpha_i \omega + \beta_i$. For $-1 \leq m, l \leq \ell - 1$, we have*

$$\begin{aligned} M_l(\partial_{h_1}, \partial_{h_2})^i M_m(\partial_{h_1}, \partial_{h_2})^j e^{-N(\mathfrak{b})N(\frac{h_2}{\alpha_{\ell-1}}\epsilon + \frac{h_1}{\alpha_{\ell-1}})} \Big|_{h=0} = \\ \partial_{h_1}^i \partial_{h_2}^j e^{-N(\mathfrak{b})N((-1)^{l+1}A_l h_1 + (-1)^{m+1}A_m h_2)} \Big|_{h=0}, \end{aligned}$$

for $M_i(x_1, x_2) = (-1)^{i+1} ((\beta_i \alpha_{\ell-1} - \alpha_i \beta_{\ell-1})x_2 + \alpha_i x_1)$.

Proof. For simplicity, let $c_i = (-1)^{i+1}(\beta_i \alpha_{\ell-1} - \alpha_i \beta_{\ell-1})$ and $d_i = (-1)^{i+1} \alpha_i$. By internal change of coordinate $(h_1, h_2) \mapsto (ah_1 + ch_2, bh_1 + dh_2)$, we have

$$(a\partial_{h_1} + b\partial_{h_2})^i (c\partial_{h_1} + d\partial_{h_2})^j f(h_1, h_2) \Big|_{h=0} = \partial_{h_1}^i \partial_{h_2}^j f(ah_1 + ch_2, bh_1 + dh_2) \Big|_{h=0}.$$

Thus

$$\begin{aligned} M_l(\partial_{h_1}, \partial_{h_2})^i M_m(\partial_{h_1}, \partial_{h_2})^j \circ e^{-N(\mathfrak{b})N(\frac{h_2}{\alpha_{\ell-1}}\epsilon + \frac{h_1}{\alpha_{\ell-1}})} \Big|_{h=0} \\ = (d_l \partial_{h_1} + c_l \partial_{h_2})^i (d_m \partial_{h_1} + c_m \partial_{h_2})^j \circ e^{-N(\mathfrak{b})N(\frac{h_2}{\alpha_{\ell-1}}\epsilon + \frac{h_1}{\alpha_{\ell-1}})} \Big|_{h=0} \\ = \partial_{h_1}^i \partial_{h_2}^j \circ e^{-N(\mathfrak{b})N(\frac{d_l h_1 + d_m h_2}{\alpha_{\ell-1}}\epsilon + \frac{c_l h_1 + c_m h_2}{\alpha_{\ell-1}})} \Big|_{h=0} \end{aligned}$$

We note that

$$\beta_{\ell-1} d_i + c_i = (-1)^{i+1} \beta_i \alpha_{\ell-1}.$$

Since $\epsilon = \alpha_{\ell-1} \omega + \beta_{\ell-1}$, we have

$$\begin{aligned} (d_l h_1 + d_m h_2) \epsilon + c_l h_1 + c_m h_2 \\ = (d_l \beta_{\ell-1} + c_l) h_1 + (d_m \beta_{\ell-1} + c_m) h_2 + (d_l h_1 + d_m h_2) \alpha_{\ell-1} \omega \\ = \alpha_{\ell-1} \left(((-1)^{l+1} \alpha_l h_1 + (-1)^{m+1} \alpha_m h_2) \omega + (-1)^{l+1} \beta_l h_1 + (-1)^{m+1} \beta_m h_2 \right) \\ = \alpha_{\ell-1} ((-1)^{l+1} A_l h_1 + (-1)^{m+1} A_m h_2) \end{aligned}$$

□

We note that

$$\begin{aligned} N(\mathfrak{b})N((-1)^{l+1}A_l h_1 + (-1)^{m+1}A_m h_2) \\ = Q((-1)^{l+1}\alpha_l h_1 + (-1)^{m+1}\alpha_m h_2, (-1)^{l+1}\beta_l h_1 + (-1)^{m+1}\beta_m h_2), \end{aligned}$$

for a binary quadratic form $Q(x, y)$ with degree 2 and i, j with $i + j = 2k$, we have

$$\partial_{h_1}^i \partial_{h_2}^j e^{-Q(h_1, h_2)} \Big|_{h=0} = (-1)^k \frac{1}{k!} \partial_{h_1}^i \partial_{h_2}^j Q(h_1, h_2)^k \Big|_{h=0}$$

Thus, from (7.6) (7.7) and Lemma 7.1, we have finished the proof of our second main theorem(Thm. 1.2).

$$\zeta(-k, \mathfrak{b}) =$$

$$\begin{aligned} & \sum_{i=0}^{\ell-1} (-1)^{i-1} L_k(\partial_{h_1}, \partial_{h_2}) Q(\alpha_i h_1 - \alpha_{i-1} h_2, \beta_i h_1 - \beta_{i-1} h_2)^k \Big|_{h=0} \\ & + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i} R_n(\partial_{h_1}, \partial_{h_2}) Q(\alpha_{i-2} h_1 + \alpha_i h_2, \beta_{i-2} h_1 + \beta_i h_2)^k \Big|_{h=0}. \end{aligned}$$

Remark 7.2. We have obtained a polynomial expression of the zeta value in variables α_i, β_i and the coefficients of the quadratic form $Q(x, y)$. For the polynomial expression, it is important to show the vanishing (7.4). In Sec. 9, there appear $\alpha_0, \alpha_{\ell-1}$ in the denominator of the vanishing expression involving the \mathcal{R} -operator. This is a crucial ingredient of the Kummer congruence and the corresponding p -adic zeta function.

8. COMPUTATION OF $\zeta(-k, \mathfrak{b})$ FOR $k = 0, 1$ AND 2

In this section, we evaluate the zeta values $\zeta(-k, \mathfrak{b})$ explicitly for small n . We express the values in terms of the continued fraction expansion $[[a_0, a_1, \dots, a_{\ell-1}]]$ of the reduced basis ω of \mathfrak{b}^{-1} .

8.1. $k=0$. For $k = 0$, the zeta value is already known by C. Meyer([27]) in terms of negative continued fraction. Using the plus-to-minus conversion formula of continued fraction

(8.1)

$$\begin{aligned} \delta &= \omega + 1 = [[a_0, a_1, \dots, a_{\ell-1}]] + 1 \\ &= ((a_0 + 2, 2, \dots, 2, a_2 + 2, 2, \dots, 2, a_4 + 2, \dots, a_{\ell-2} + 2, 2, \dots, 2)) \\ &= ((b_0, b_1, \dots, b_{m-1})) \end{aligned}$$

one obtains the result in positive continued fraction.

In our approach, we begin with the expression using positive continued fraction as a special case of Thm.1.2:

$$\zeta(0, \mathfrak{b}) = \frac{B_2}{2} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i}$$

Since $B_2 = 1/6$ and ℓ is the even period of the continued fraction, this reduces to

$$(8.2) \quad \zeta(0, \mathfrak{b}) = \frac{1}{12} \sum_{i=0}^{\ell-1} (-1)^i a_i$$

Via (8.1), one recovers the result of Meyer:

$$\zeta(0, \mathfrak{b}) = \frac{1}{12} \sum_{i=0}^{m-1} (b_i - 3),$$

where b_i is the i -th term of the negative continued fraction.

Remark 8.1. Note that using the positive continued fraction, we have an alternating sum for the zeta value. Consequently, one sees directly the vanishing of $\zeta(0, \mathfrak{b})$ when the actual period of the positive continued fraction of ω is odd (equivalently, if the fundamental unit is not totally positive).

8.2. $k=1$ and 2 . For $Q(x_1, x_2) = N(\mathfrak{b})N(x_1\omega + x_2)$, let L_i , M_i and N_i be defined as in

$$Q(\alpha_i h_1 - \alpha_{i-1} h_2, \beta_i h_1 - \beta_{i-1} h_2) = L_i h_1^2 + M_i h_1 h_2 + N_i h_2^2$$

Similarly, \tilde{L}_i , \tilde{M}_i and \tilde{N}_i are defined as follows:

$$Q(\alpha_{i-2} h_1 + \alpha_i h_2, \beta_{i-2} h_1 + \beta_i h_2) = \tilde{L}_i h_1^2 + \tilde{M}_i h_1 h_2 + \tilde{N}_i h_2^2$$

Then the special value at $s = -1$ is computed out as follows:

$$\begin{aligned} \zeta(-1, \mathfrak{b}) &= \sum_{i=0}^{\ell-1} (-1)^{i-1} \frac{B_2^2}{4} M_i + \frac{B_4}{4!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i} (2\tilde{L}_i + \tilde{M}_i + 2\tilde{N}_i) \\ &= \frac{1}{720} \sum_{i=0}^{\ell-1} (-1)^{i-1} (5M_i + a_{\ell-i} (2\tilde{L}_i + \tilde{M}_i + 2\tilde{N}_i)) \end{aligned}$$

Similarly for $s = -2$,

$$\begin{aligned} \zeta(-2, \mathfrak{b}) &= \\ &= \frac{1}{15120} \sum_{i=0}^{\ell-1} (-1)^i (21M_i(N_i + L_i) + 2a_{\ell-i} (6\tilde{L}_i^2 + 3\tilde{L}_i\tilde{M}_i + \tilde{M}_i^2 + 2\tilde{L}_i\tilde{N}_i + 3\tilde{M}_i\tilde{N}_i + 6\tilde{N}_i^2)) \end{aligned}$$

This should be compared with the expression obtained using negative continued fraction in [12] and also [34]. They considered the zeta function of the following quadratic form in view of negative continued fraction:

$$Q'(x_1, x_2) := N(\mathfrak{b})N(x_1\delta + x_2)$$

for $\delta = \omega + 1$ (See (8.1) for negative continued fraction). Let A_i be the lattice points of the component of the Klein polyhedron of \mathfrak{b}^{-1} in the 1st

quadrant with normalization: $A_0 = 1, A_{-1} = \delta$ and the 1st coordinate of A_i increasing according to i . Then we associate a lattice vector (p_k, q_k) to A_k for $A_k = -p_k A_{-1} + q_k A_0$. p_k and q_k are obtain from the reduced fraction of the truncation after k the of the negative continued fraction $\delta = ((b_0, b_1, \dots, b_m))$:

$$\frac{q_k}{p_k} = (b_0, \dots, b_{k-1})$$

(This is the last line of pp.18 of [12], where $\frac{p_k}{q_k}$ should be corrected to $\frac{q_k}{p_k}$ as we just wrote above). Similarly, L'_i, M'_i, N'_i and $\tilde{L}'_i, \tilde{M}'_i, \tilde{N}'_i$ are defined as the coefficients of quadratic forms:

$$Q'(-p_{i-1}h_1 - p_i h_2, q_{i-1}h_1 + q_i h_2) = L'_i h_1^2 + M'_i h_1 h_2 + N'_i h_2^2$$

and

$$Q'(-p_{i-1}h_1 - p_{i+1}h_2, q_{i-1}h_1 + q_{i+1}h_2) = \tilde{L}'_i h_1^2 + \tilde{M}'_i h_1 h_2 + \tilde{N}'_i h_2^2.$$

In this setting, Garoufalidis-Pommersheim([12]) obtained:

$$\zeta(-1, \mathfrak{b}) = \frac{1}{720} \sum_{i=0}^{m-1} (5M'_i + b_i(-2\tilde{L}'_i + \tilde{M}'_i - 2\tilde{N}'_i))$$

and

$$\begin{aligned} \zeta(-2, \mathfrak{b}) = & \frac{1}{15120} \sum_{i=0}^{m-1} (-21M'_i(L'_i + N'_i) \\ & + 2b_i(6\tilde{L}'_i{}^2 - 3\tilde{L}'_i\tilde{M}'_i + 2\tilde{L}'_i\tilde{N}'_i + \tilde{M}'_i{}^2 - 3\tilde{M}'_i\tilde{N}'_i + 6\tilde{N}'_i{}^2)). \end{aligned}$$

It should be also compared with Zagier's result(eg. for $k = 1$) in [34]:

$$\zeta(-1, \mathfrak{b}) = \frac{1}{720} \sum_{i=0}^{m-1} (-2N_i b_i^3 + 3M_i b_i^2 - 6L_i b_i + 5M_i)$$

Remark 8.2. *If we use the formula for zeta values using negative continued fractions as is made by Garoufalidis-Pommersheim and Zagier, one can still obtain polynomial behavior in a family similar to Sec. 10 after the uniformity of the negative continued fractions in the family. But as is known, there is hardly a direct arithmetic property(eg. regulator) associated to the negative continued fractions. Actually in [20], [21], the formula for negative continued fractions, which is developed by Yamamoto([33]) and Zagier([34], [35]), is used after conversion of positive continued fraction into negative one. The formula using positive continued fraction simplifies this unnecessary step and justifies the reason of our earlier results.*

9. VANISHING PART

Now, it remains to show the vanishing of (7.4)

$$\mathcal{R} \circ \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 \Big|_{h=0} = 0.$$

This part is crucial in expressing the zeta values at nonpositive integers as polynomials of its argument coming from terms of continued fractions and the coefficients of quadratic forms. The vanishing has been already observed in related works by Zagier([34]) and Garoufalidis-Pommersheim([12]) in different settings. In [12], only the vanishing is mentioned without clear proof. In this section, we will recycle some notions and ideas from [34].

It suffices to show the vanishing of the following, which equals the above up to multiplication by a constant.

$$(9.1) \quad \left(-a_\ell R_k(M_{-2}, M_0)(\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} - \partial_{h_1} M_0^{2k+1}(\partial_{h_1}, \partial_{h_2}) \right. \\ \left. + \partial_{h_2} M_{\ell-2}^{2n+1}(\partial_{h_1}, \partial_{h_2}) \right) \circ \int_{\sigma(h_1, h_2)} e^{-Q(x_1, x_2)} dx_1 dx_2 \Big|_{h=0}$$

As $M_0 = \beta_{\ell-1} x_2 - x_1$ and $M_{\ell-2} = x_2 - \alpha_{\ell-2} x_1$, we have

$$(9.2) \quad x_1 M_0^{2k+1} + x_2 M_{\ell-2}^{2k+1} \\ = 2x_1^{2k+2} + \sum_{i=1}^{2k+1} (-1)^i \binom{2k+1}{i} (\beta_{\ell-1}^i + \alpha_{\ell-2}^i) x_2^i x_1^{2k+2-i}.$$

Applying $\partial_{h_1}^{2k+2}$ to (7.5), we obtain

$$(9.3) \quad \alpha_{\ell-1} \partial_{h_1}^{2k+2} \int_{-\frac{h_2}{\alpha_{\ell-1}}}^{\infty} \int_{-\frac{h_1}{\alpha_{\ell-1}}}^{\infty} e^{-N(b)N(\epsilon y_2 + y_1)} dy_1 dy_2 \Big|_{h=0} \\ = \frac{1}{\alpha_{\ell-1}} \int_0^{\infty} \partial_{h_1}^{2k+1} e^{-N(b)N(\epsilon \frac{y_2}{\alpha_{\ell-1}} - \frac{h_1}{\alpha_{\ell-1}})} \Big|_{h_1=0} dy_2$$

If we write $P(x_1, x_2) = \frac{N(b)}{\alpha_{\ell-1}^2}(x_2^2 + (\epsilon + \epsilon')x_1x_2 + x_1^2)$, using (9.2)-(9.3), one can simplify the 2nd half of (9.1):

(9.4)

$$\begin{aligned} & \left(-\partial_{h_1} M_0(\partial_{h_1}, \partial_{h_2})^{2k+1} + \partial_{h_2} M_{\ell-2}(\partial_{h_1}, \partial_{h_2})^{2k+1} \right) \circ \int_{\sigma(h_1, h_2)} e^{-Q(x_1, x_2)} dx_1 dx_2 \Big|_{h=0} \\ &= \frac{1}{\alpha_{\ell-1}} \sum_{i=1}^{2k+1} (-1)^i \binom{2k+1}{i} (\beta_{\ell-1}^i + \alpha_{\ell-2}^i) \partial_{h_2}^{i-1} \partial_{h_1}^{2k+1-i} \circ e^{-P(h_1, h_2)} \Big|_{h=0} \\ & \quad + \frac{2}{\alpha_{\ell-1}} \int_0^\infty \partial_{x_1}^{2k+1} e^{-P(-x_1, x_2)} \Big|_{x_1=0} dx_2 \end{aligned}$$

From $M_{-2} = (a_0 \alpha_{\ell-1} + \beta_{\ell-1})x_2 - x_1$ and $M_0 = \beta_{\ell-1}x_2 - x_1$, we have

$$R_k(M_{-2}, M_0) = \sum_{i=0}^{2k+1} (-1)^{i+1} \binom{2k+1}{i} \frac{(a_0 \alpha_{\ell-1} + \beta_{\ell-1})^i - \beta_{\ell-1}^i}{a_0 \alpha_{\ell-1}} x_2^{i-1} x_1^{2k+1-i}.$$

From (9.4) and (9.5), we obtain the following lemma:

Lemma 9.1. *Let $P(x_1, x_2) = \frac{N(b)}{\alpha_{\ell-1}^2}(x_2^2 + (\epsilon + \epsilon')x_1x_2 + x_1^2)$. Then we have*

$$\begin{aligned} & \left(-a_\ell R_k(M_{-2}, M_0)(\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} - \partial_{h_1} M_0^{2k+1}(\partial_{h_1}, \partial_{h_2}) \right. \\ & \quad \left. + \partial_{h_2} M_{\ell-2}^{2n+1}(\partial_{h_1}, \partial_{h_2}) \right) \circ \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 \Big|_{h=0} \\ &= \frac{1}{\alpha_{\ell-1}} \sum_{i=0}^{2k} (-1)^{i+1} \binom{2k+1}{i+1} ((a_0 \alpha_{\ell-1} + \beta_{\ell-1})^{i+1} + (\alpha_{\ell-2})^{i+1}) \partial_{h_2}^i \partial_{h_1}^{2k-i} \circ e^{-P(h)} \Big|_{h=0} \\ & \quad + \frac{2}{\alpha_{\ell-1}} \int_0^\infty \partial_{x_1}^{2k+1} e^{-P(-x_1, x_2)} \Big|_{x_1=0} dx_2. \end{aligned}$$

Lemma 9.2. *For the totally positive fundamental unit $\epsilon > 1$, we have*

$$\epsilon + \epsilon' = a_0 \alpha_{\ell-1} + \beta_{\ell-1} + \alpha_{\ell-2}.$$

Proof. We note that

$$\delta := -\frac{1}{\omega'} = [[a_{\ell-1}, a_{\ell-2}, \dots, a_0]].$$

Thus

$$\delta = \begin{pmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} = \frac{\alpha_\ell \delta + \alpha_{\ell-1}}{\beta_\ell \delta + \beta_{\ell-1}}.$$

And we have

$$\alpha_{\ell-1} \omega^2 - (\alpha_\ell - \beta_{\ell-1}) \omega - \beta_\ell = 0.$$

Finally we have

$$\omega + \omega' = \frac{\alpha_\ell - \beta_{\ell-1}}{\alpha_{\ell-1}}.$$

$$\epsilon + \epsilon' = \alpha_{\ell-1}(\omega + \omega') + 2\beta_{\ell-1} = \alpha_\ell + \beta_{\ell-1}$$

Thus

$$\frac{\epsilon + \epsilon' - \beta_{\ell-1} - \alpha_{\ell-2}}{\alpha_{\ell-1}} = a_0.$$

□

From Lemma 9.2, if we let $a_0\alpha_{\ell-1} + \beta_{\ell-1} = -a$, $\alpha_{\ell-2} = -b$ and $\frac{N(b)}{\alpha_{\ell-1}^2} = A$ then we find that $\epsilon + \epsilon' = -(a + b)$. Hence one can rewrite Lemma 9.1 as follows:

(9.6)

$$\begin{aligned} & \sum_{i=0}^{2k} (-1)^{i+1} \binom{2k+1}{i+1} ((a_0\alpha_{\ell-1} + \beta_{\ell-1})^{i+1} + (\alpha_{\ell-2})^{i+1}) \partial_{h_2}^i \partial_{h_1}^{2k-i} \circ e^{-P(h_1, h_2)} \Big|_{h=0} \\ & \quad + 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-P(-x_1, x_2)} \Big|_{x_1=0} dx_2 \\ & = \sum_{i=0}^{2k} \binom{2k+1}{i+1} (a^{i+1} + b^{i+1}) \partial_{h_2}^i \partial_{h_1}^{2k-i} \circ e^{-A(h_2^2 - (a+b)h_1h_2 + h_1^2)} \Big|_{h=0} \\ & \quad + 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_2^2 + (a+b)x_1x_2 + x_1^2)} \Big|_{x_1=0} dx_2. \end{aligned}$$

Hence it remains to show vanishing of the right hand side of (9.6):

$$\begin{aligned} (\star) \quad & \sum_{i=0}^{2k} \binom{2k+1}{i+1} (a^{i+1} + b^{i+1}) \partial_{h_2}^i \partial_{h_1}^{2k-i} \circ e^{-A(h_2^2 - (a+b)h_1h_2 + h_1^2)} \Big|_{h=0} \\ & + 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_2^2 + (a+b)x_1x_2 + x_1^2)} \Big|_{x_1=0} dx_2. \end{aligned}$$

For the proof, we introduce $f_k(\alpha, \beta, \gamma)$ and $d_{r,k}(\alpha, \beta, \gamma)$ as follows:

$$(9.7) \quad \int_0^\infty \partial_{x_1}^{2k+1} e^{-(\alpha x_2^2 + \beta x_1 x_2 + \gamma x_1^2)} \Big|_{x_1=0} dx_2 = -\frac{(2k+1)! f_k(\alpha, \beta, \gamma)}{2\gamma^{k+1}}$$

$$(9.8) \quad \sum_{i=0}^{2k} d_{i,2k-i}(\alpha, \beta, \gamma) x_1^i x_2^{2k-i} = (\alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2)^k$$

These numbers are originally appeared in [34]. One should see that $f_k(\alpha, \beta, \gamma)$ is odd function w.r.t. β :

$$f_k(\alpha, \beta, \gamma) + f_k(\alpha, -\beta, \gamma) = 0.$$

One can identify $d_{i,2k-i}(\alpha, -\beta, \gamma)$ in the following expression:

$$\begin{aligned} & \left. \partial_{x_1}^i \partial_{x_2}^{2k-i} e^{-(\alpha x_1^2 - \beta x_1 x_2 + \gamma x_2^2)} \right|_{(x_1, x_2) = (0, 0)} \\ &= \frac{(-1)^k}{k!} \partial_{x_1}^i \partial_{x_2}^{2k-i} (\alpha x_1^2 - \beta x_1 x_2 + \gamma x_2^2)^k = \frac{(-1)^k}{k!} i! (2k-i)! d_{i,2k-i}(\alpha, -\beta, \gamma). \end{aligned}$$

From this, one can rewrite the 1st line of (\star) as

$$\begin{aligned} (9.9) \quad & \sum_{i=0}^{2k} \binom{2k+1}{i+1} (a^{i+1} + b^{i+1}) \partial_{h_2}^i \partial_{h_1}^{2k-i} \circ e^{-A(h_2^2 - (a+b)h_1 h_2 + h_1^2)} \Big|_{h=0} \\ &= \frac{(-1)^k}{k!} (2k+1)! \sum_{i=0}^{2k} \frac{a^{i+1} + b^{i+1}}{i+1} d_{i,2k-i}(A, -A(a+b), A) \end{aligned}$$

The 2nd line of (\star) is, from the definition of $f_k(\alpha, \beta, \gamma)$,

$$(9.10) \quad 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_2^2 + (a+b)x_1 x_2 + x_1^2)} \Big|_{x_1=0} dx_2 = -(2k+1)! \frac{f_k(A, A(a+b), A)}{A^{k+1}}.$$

Now, we are going to use an identity relating $f_k(\alpha, \beta, \gamma)$ and $d_{i,2k-i}(\alpha, -\beta, \gamma)$ due to Zagier:

Lemma 9.3 (Zagier(Prop. 4 of [34])). *For a real number λ , we have*

$$\begin{aligned} & f_k(\alpha, \beta, \gamma) + f_k(\gamma, 2\lambda\gamma - \beta, \lambda^2\gamma - \lambda\beta + \alpha) \\ &= 2 \frac{(-1)^k}{k!} \gamma^{k+1} \sum_{i=0}^{2k} d_{i,2k-i}(\alpha, -\beta, \gamma) \frac{\lambda^{i+1}}{i+1}. \end{aligned}$$

If we put $\alpha = A, \beta = A(a+b), \gamma = A$ and $\lambda = a$ (resp. $\lambda = b$) into the above, we obtain

$$\begin{aligned} & f_k(A, A(a+b), A) + f_k(A, A(a-b), A(-ab+1)) \\ &= 2 \frac{(-1)^k}{k!} A^{k+1} \sum_{i=0}^{2k} d_{i,2k-i}(A, -A(a+b), A) \frac{a^{i+1}}{i+1} \end{aligned}$$

and

$$\begin{aligned} & f_k(A, A(a+b), A) + f_k(A, A(b-a), A(-ab+1)) \\ &= 2 \frac{(-1)^k}{k!} A^{k+1} \sum_{i=0}^{2k} d_{i,2k-i}(A, -A(a+b), A) \frac{b^{i+1}}{i+1}. \end{aligned}$$

As f_k is odd function of its 2nd argument, summing the above two equations, we have

$$\frac{f_k(A, A(a+b), A)}{A^{k+1}} = \frac{(-1)^k}{k!} \sum_{i=0}^{2k} d_{i,2k-i}(A, -A(a+b), A) \frac{a^{i+1} + b^{i+1}}{i+1}.$$

This identifies (9.9) and (9.10) up to sign.

Therefore we conclude the vanishing of (\star) .

10. APPLICATION: POLYNOMIAL BEHAVIOR OF ZETA VALUES AT NONPOSITIVE INTEGERS IN FAMILY

Until now, we developed a way to compute the partial zeta values at nonpositive integers for a real quadratic field with a fixed ideal \mathfrak{b} via the shape of the continued fraction of ω for $\mathfrak{b}^{-1} = [1, \omega]$. We will apply this method to certain families of real quadratic fields to prove the main theorem of this paper (Thm. 1.1). We deal with the same family of real quadratic fields with ideals fixed as in our earlier works ([20], [21]). In our previous works, the partial Hecke's L -values and the partial zeta values of a ray class ideal at $s = 0$ are investigated for family of real quadratic fields. We showed that the values in the family is given by a quasi-polynomial in variable n which is the index of the family of real quadratic fields considered. If the conductor is trivial, so that we consider ideal classes, the values behave actually in a polynomial. This method was originally observed by Biró and has been main ingredient to solve class number problems of the real quadratic fields in the family without relying on the Riemann hypothesis (cf. [2], [3], [5], [6], [7]). Here we deal with the case when the conductor is trivial. Thus we have strict polynomial instead of quasi-polynomials.

We generalize the result on the partial zeta values at $s = 0$ to every nonpositive integer s when the conductor is trivial. This means we consider partial zeta function of ideal classes instead of ray classes. So the scope of partial zeta functions we consider here is narrower than the previous. But the same method must be applicable to ray class partial zeta functions. In this case by the same reason quasi-polynomials are appearing instead of polynomials to give the zeta values at a given nonpositive integer for the same family of ideals. Again this will answer the same for the partial Hecke L -values at arbitrary non-positive integers.

Recall the conditions on the family (K_n, \mathfrak{b}_n) indexed by $n \in N$ for a subset N of \mathbb{N} . $\mathfrak{b}_n^{-1} = [1, \omega(n)]$ for a reduced element $\omega(n) \in K_n$ and

$$\omega(n) = [[a_0(n), a_1(n), \dots, a_{r-1}(n)]]$$

for polynomials $a_i(x) \in \mathbb{Z}[x]$ and the quadratic form $N(\mathfrak{b}_n)(x\omega(n) + y)(x\omega(n)' + y)$ associated with \mathfrak{b}_n is expressed as

$$b_1(n)x^2 + b_2(n)xy + b_3(n)y^2$$

for polynomials $b_i(x) \in \mathbb{Z}[x]$.

Proof of Thm.1.1. Applying Thm.1.2 to the family considered, we have

$$\begin{aligned} \zeta(-k, \mathfrak{b}_n) = & \sum_{i=0}^{\ell-1} (-1)^{i-1} L_k(\partial_{h_1}, \partial_{h_2}) Q(\alpha_i(n)h_1 - \alpha_{i-1}(n)h_2, \beta_i(n)h_1 - \beta_{i-1}(n)h_2)^k \\ & + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i}(n) R_k(\partial_{h_1}, \partial_{h_2}) Q(\alpha_{i-2}(n)h_1 + \alpha_i(n)h_2, \beta_{i-2}(n)h_1 + \beta_i(n)h_2)^k. \end{aligned}$$

Since $Q(-)$ is a quadratic form and $a_i(n), b_i(n), \alpha_i(n), \beta_i(n)$ are polynomials, it is clear that $\zeta(-k, \mathfrak{b}_n)$ is a polynomial in n .

Notice that

$$\deg \alpha_i \geq \deg \beta_i$$

and

$$\deg \alpha_i \geq \deg \alpha_{i-1}.$$

Thus, the highest degree term comes from the summand with $i = \ell - 1$. Putting altogether, we obtain the denominator C_k as well as the degree $m = kC + D$ for the explicitly given C, D . \square

Remark 10.1. One should notice the independence of n of the denominator C_k of $\zeta(-k, \mathfrak{b}_n)$. A priori this is invariant in the family. It is important to control the denominator to interpolate the associated p -adic zeta function from the values at negative integers (cf. [9], [10] and [24]).

For the rest of the paper, for a number field K , let us denote the ring of integers by O_K .

Example 10.2. Consider the family $(K_n = \mathbb{Q}(\sqrt{n^2 + 2}), \mathfrak{b}_n = O_{K_n})$. Then

$$\mathfrak{b}_n^{-1} = O_{K_n} = [1, \omega_n]$$

for $\omega_n = [[2n, n]]$.

Then we have

$$\begin{aligned} \zeta(0, \mathfrak{b}_n) &= \frac{n}{12} \\ \zeta(-1, \mathfrak{b}_n) &= -\frac{19n}{360} + \frac{n^3}{40} \\ \zeta(-2, \mathfrak{b}_n) &= \frac{2n}{45} - \frac{n^3}{945} - \frac{23n^5}{1890} \\ \zeta(-3, \mathfrak{b}_n) &= -\frac{2159n}{25200} + \frac{137n^3}{25200} - \frac{59n^5}{840} + \frac{3n^7}{56} \\ \zeta(-4, \mathfrak{b}_n) &= \frac{68n}{231} - \frac{797n^3}{6930} + \frac{689n^5}{1155} + \frac{134n^7}{1155} - \frac{2878n^9}{10395} \\ \zeta(-5, \mathfrak{b}_n) &= -\frac{11947883n}{7567560} + \frac{29660563n^3}{22702680} - \frac{26073083n^5}{5675670} - \frac{7603n^7}{2310} - \frac{145933n^9}{135135} + \frac{351719n^{11}}{135135} \end{aligned}$$

Example 10.3. Let $K_n = \mathbb{Q}(\sqrt{16n^4 + 32n^3 + 24n^2 + 12n + 3})$ and $\mathfrak{b}_n = O_{K_n}$. Then $\mathfrak{b}_n^{-1} = O_{K_n} = [1, \omega_n]$ for $\omega_n = [[8n^2 + 8n + 2, 2n + 1]]$.

$$\begin{aligned}
\zeta(0, \mathfrak{b}_n) &= \frac{1}{12} + \frac{n}{2} + \frac{2n^2}{3} \\
\zeta(-1, \mathfrak{b}_n) &= -\frac{7}{72} - \frac{13n}{20} - \frac{11n^2}{9} + \frac{n^3}{45} + \frac{34n^4}{15} + \frac{104n^5}{45} + \frac{32n^6}{45} \\
\zeta(-2, \mathfrak{b}_n) &= \frac{503}{2520} + \frac{2773n}{1260} + \frac{8473n^2}{945} + \frac{13009n^3}{945} - \frac{6898n^4}{945} - \frac{360n^5}{7} - \frac{6208n^6}{105} \\
&\quad - \frac{3328n^7}{315} + \frac{25472n^8}{945} + \frac{18944n^9}{945} + \frac{4096n^{10}}{945} \\
\zeta(-3, \mathfrak{b}_n) &= -\frac{823}{840} - \frac{7762n}{525} - \frac{193469n^2}{2100} - \frac{309377n^3}{1050} - \frac{232553n^4}{525} + \frac{143188n^5}{1575} + \frac{2707724n^6}{1575} \\
&\quad + \frac{5759672n^7}{1575} + \frac{7377392n^8}{1575} + \frac{7421248n^9}{1575} + \frac{147072n^{10}}{35} + \frac{4862464n^{11}}{1575} + \frac{830464n^{12}}{525} \\
&\quad + \frac{249856n^{13}}{525} + \frac{32768n^{14}}{525} \\
\zeta(-4, \mathfrak{b}_n) &= \frac{106613}{11880} + \frac{262407n}{1540} + \frac{1957759n^2}{1386} + \frac{23147174n^3}{3465} + \frac{203979376n^4}{10395} + \frac{365417032n^5}{10395} \\
&\quad + \frac{257724232n^6}{10395} - \frac{764543312n^7}{10395} - \frac{1238665888n^8}{3465} - \frac{3134586496n^9}{3465} - \frac{3314036224n^{10}}{2079} \\
&\quad - \frac{4177427456n^{11}}{2079} - \frac{880111616n^{12}}{495} - \frac{2179907584n^{13}}{2079} - \frac{27426816n^{14}}{77} \\
&\quad - \frac{86638592n^{15}}{3465} + \frac{22151168n^{16}}{693} + \frac{12058624n^{17}}{945} + \frac{16777216n^{18}}{10395} \\
\zeta(-5, \mathfrak{b}_n) &= -\frac{4617527}{34398} - \frac{52647823n}{17199} - \frac{71296254191n^2}{2270268} - \frac{22288517357n^3}{115830} - \frac{2248765926611n^4}{2837835} \\
&\quad - \frac{202240251208n^5}{85995} - \frac{1639280941052n^6}{315315} - \frac{7365379306328n^7}{945945} - \frac{45593045200n^8}{51597} \\
&\quad + \frac{124808351658752n^9}{2837835} + \frac{501230433622144n^{10}}{2837835} + \frac{1216530615292672n^{11}}{2837835} + \frac{421424974443008n^{12}}{567567} \\
&\quad + \frac{2733316068964352n^{13}}{2837835} + \frac{301312337145856n^{14}}{315315} + \frac{694067022135296n^{15}}{945945} + \frac{416253219504128n^{16}}{945945} \\
&\quad + \frac{84428067700736n^{17}}{405405} + \frac{223080016510976n^{18}}{2837835} + \frac{13420435865600n^{19}}{567567} + \frac{15443400065024n^{20}}{2837835} \\
&\quad + \frac{481111834624n^{21}}{567567} + \frac{185488900096n^{22}}{2837835}
\end{aligned}$$

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